Boundaries, Green's formulae

and harmonic functions

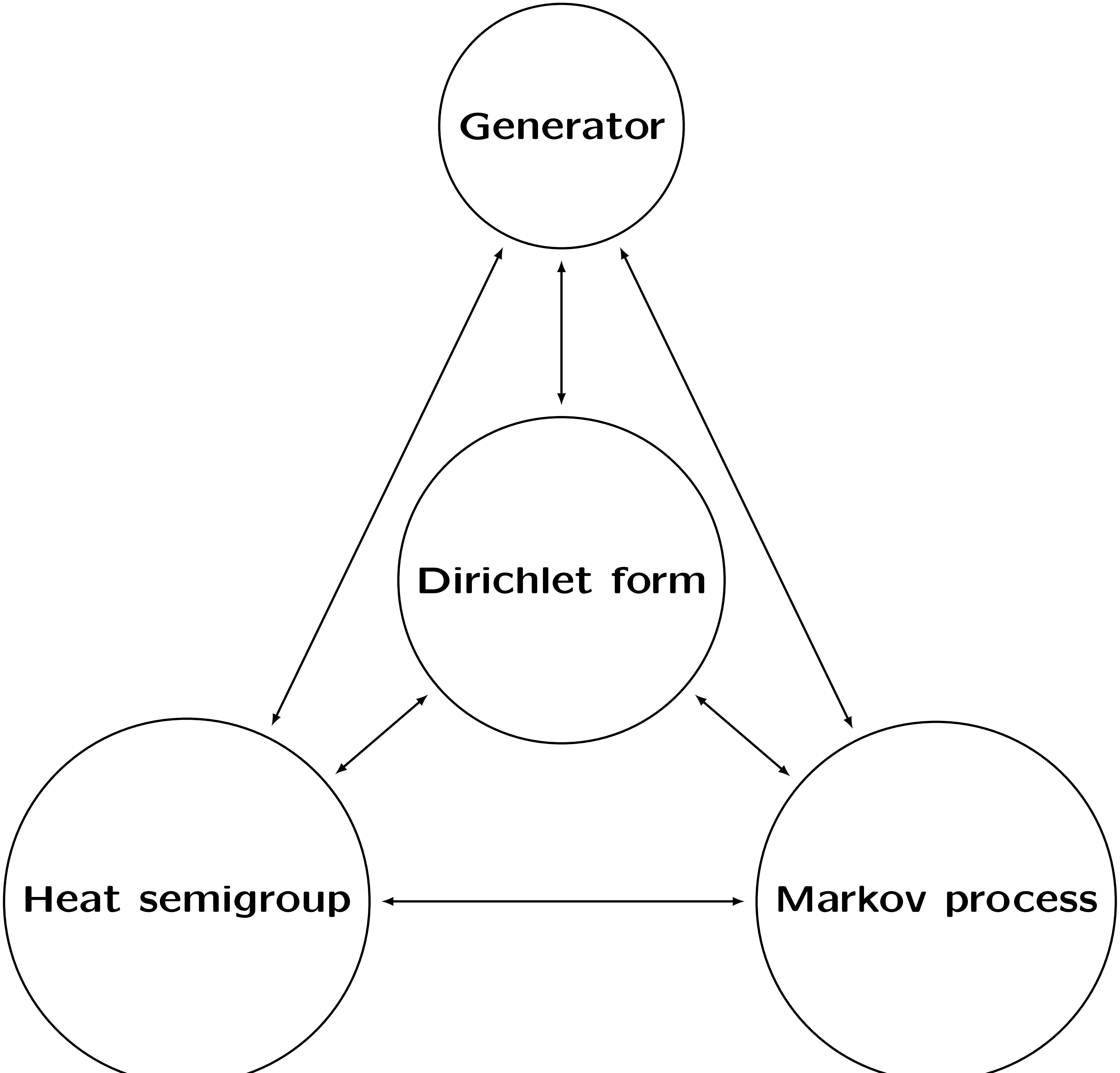
for graphs and Dirichlet spaces

Matthias Keller (Potsdam) and Daniel Lenz (Jena)
with Marcel Schmidt (Jena)

To set the stage consider following four related objects in $\mathbb{R}^n$:

- Quadratic form: $Q(f) = \int\limits_{\mathbb{R}^n} |\nabla f|^2 dx$
- Laplace operator: $L = -\sum_{i=1}^{n} \partial_i^2$ s.t. $\langle Lf, f\rangle = Q(f,f)$
- Heat semigroup: $e^{-tL}f(x) = \int_{\mathbb{R}^d} f(y) \frac{1}{(4\pi t)^{n/2}} e^{-\frac{|x-y|^2}{4t}} dy$
- Brownian motion: $(X_t, \mathcal{F}_t, \Omega, \mathbb{E})$ with $\mathbb{E}_x(f(X_t)) = e^{-tL}f(x)$.

This is just a special case of...



Generator
Dirichlet form
Heat semigroup
Markov process

Definition. A *Dirichlet space* is a locally compact separable metric measure space (X, m) together with a closed quadratic form (Q, D) on $L^2(X, m)$ that satisfies the Markov property

$$Q(\min\{1, f\}) \leq Q(f), \quad f \in L^2(X, m).$$

Condition may seem inconspicuous...

But this setting

- **covers manifolds and** metric measure spaces as well as discrete **graphs** and quantum graphs and fractals,

- **connects analysis and stochastic** by giving analytic description of symmetric Markov processes, Each Dirichlet space comes with

- selfadjoint operator, called the generator (Laplacian),

- Markov process (Brownian motion),

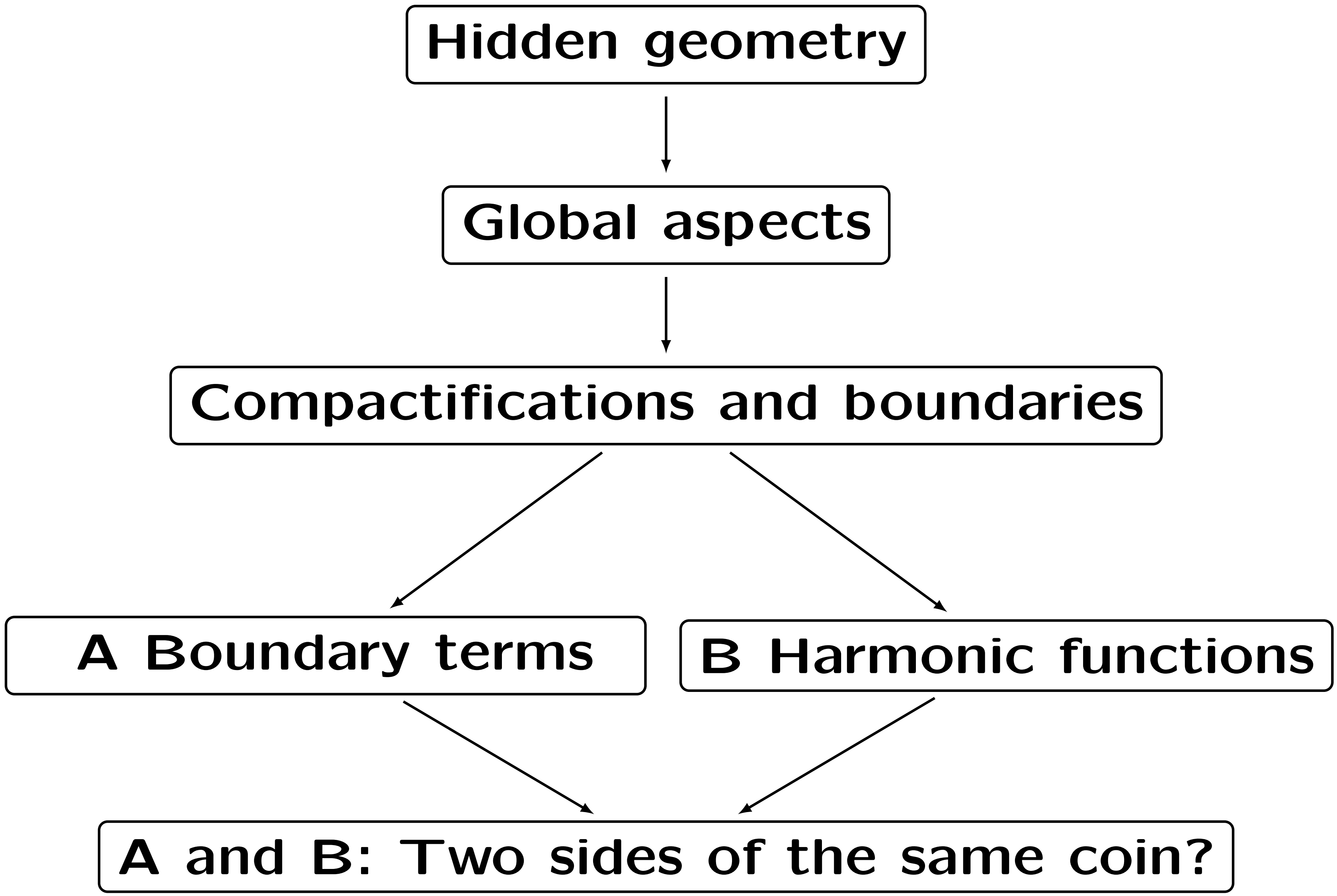
- semigroup (heat semigroup).

Spectral theory of the generator and behavior of the heat semigroup **reflect various aspects of the geometry** as explored by the Markov process.

Proposal deals with **global aspects of geometry** in the context of Dirichlet spaces.

Main focus is to capture these aspects via certain **compactifications** and the associated **boundaries**.

There are two (related) directions: **boundary terms** and **harmonic functions.**


Hidden geometry
Global aspects
Compactifications and boundaries
A Boundary terms
B Harmonic functions
A and B: Two sides of the same coin?

Something concrete:
The Royden boundary and canonically compactifiable graphs.

Consider a *graph* i.e. $b : X \times X \to [0, \infty)$ symmetric with

$$\sum_{y \in X} b(x, y) < \infty, \qquad x \in X.$$

Associated *quadratic form* $\mathcal{Q} : C(X) \to [0, \infty]$

$$\mathcal{Q}(f) = \frac{1}{2} \sum_{x,y \in X} b(x, y)(f(x) - f(y))^2.$$

Associated *functions of finite energy*

$$\mathcal{D} = \{f \in C(X) \mid \mathcal{Q}(f) < \infty\}.$$

Note: Since $|\min\{1, a\} - \min\{1, b\}| \leq |a - b|$ we have indeed

$$\mathcal{Q}(\min\{1, f\}) \leq \mathcal{Q}(f).$$

The Royden compactification

The space

$$\mathcal{D} \cap \ell^{\infty}$$

is an involutive unital algebra.

Gelfand theory for C^*-algebras gives

$$\mathcal{A} := \overline{\mathcal{D} \cap \ell^{\infty}}^{\|\cdot\|_{\infty}} \cong C(K)$$

with K compact Hausdorff space of multiplicative functionals on $\mathcal{A}$:

- $X \hookrightarrow K$ dense via $x \mapsto \delta_x$, i.e., $\delta_x(f) = f(x)$, $f \in \mathcal{D} \cap \ell^{\infty}$.
- Every $f \in \mathcal{D} \cap \ell^{\infty}$ has continuous extension to K.
- $\mathcal{D}$ separates the points of K.

Royden boundary

$$\partial X = K \setminus X.$$

Special case: A graph is called *canonically compactifiable* if

$$\mathcal{D} \subseteq \ell^\infty.$$

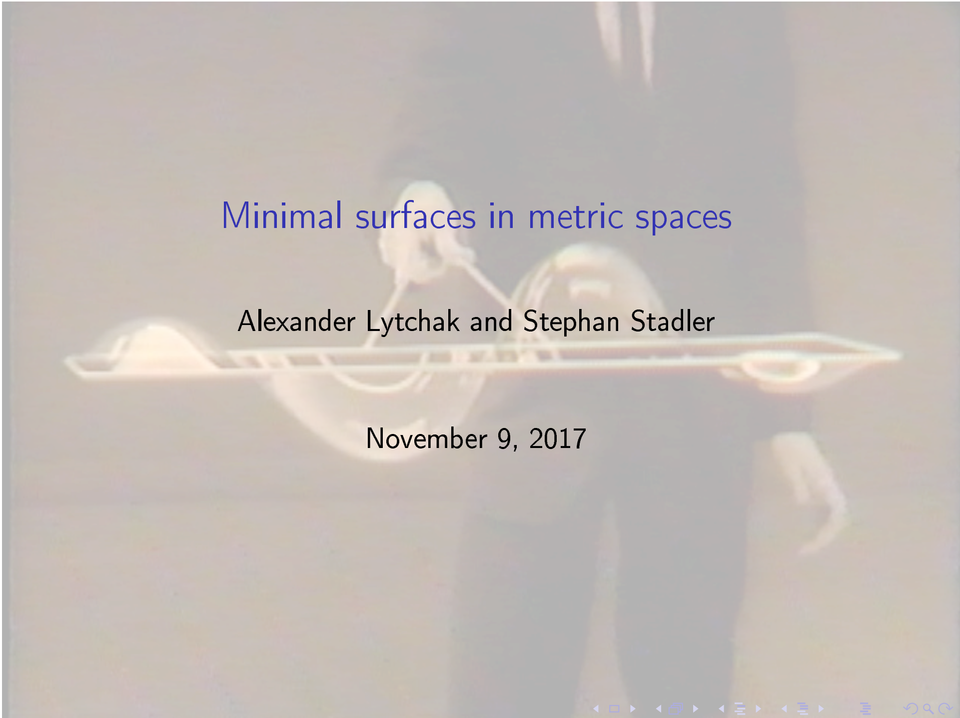
This is nice because:

- ∂X is a metric boundary.
- Dirichlet problem uniquely solvable.
- All Laplacian to $\mathcal{Q}$ on $\mathcal{D} \cap \ell^2$ have purely discrete spectrum.
- One can characterize all Dirichlet forms by boundary forms.
- ...

$\rightsquigarrow$ discrete analogues to precompact manifolds with nice boundary.

A bouqet of questions within the project:

- When is ∂X is a metric boundary?
- Boundary Harnack inequalities.
- Selfadjoint extensions and boundary conditions.
- Green's formulae and boundary terms.
- Liouville theorems.
- Harmonic functions of polynomial growth.
- Nodal domains.



Minimal surfaces in metric spaces

Alexander Lytchak and Stephan Stadler

November 9, 2017

Overview

1. Solution of Plateau's problem

2. Applications: found and to be found

How it all started

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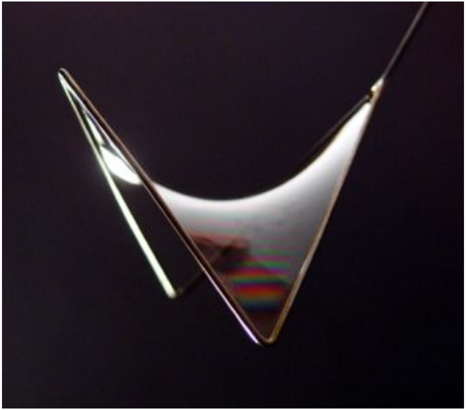

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Theorem (Douglas-Rado 1930)

Let Γ be a smooth, simple closed curve in $\mathbb{R}^n$. Then Γ can be spanned by a smooth disc $u : \bar{D} \to \mathbb{R}^n$ of minimal area. Moreover, u can be chosen (weakly) conformal.

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- Extensions to Riemannian manifolds, some 3-D Finsler spaces, some Alexandrov spaces (Morrey, v.d. Mosel-Overath, Mese).

Extensions to metric spaces

Theorem (L.-Wenger'15)

Let X be a proper metric space and $\Gamma \subset X$ such that $\Lambda(\Gamma) \neq \emptyset$. Then there exists an area minimizer in $\Lambda(\Gamma)$. Moreover it comes with a canonical parametrization.

Quadratic isoperimetric inequalities

Definition

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Theorem (L.-Wenger'15,17)

Let X, Γ be as above. If X satisfies a quadratic isoperimetric inequality, then any minimal disc is a geometric object with controllable intrinsic geometry.

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- Use this tool!

Applications to non-positively curved spaces

Theorem (Isoperimetric characterization, L.-Wenger'16)

Let X be a proper geodesic space. Then X is CAT(0) if and only if X admits a quadratic isoperimetric inequality with constant $\frac{1}{4\pi}$.

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Theorem (Fary-Milnor, S.'17)

Let X be a CAT(0) space. Let Γ be a Jordan curve in X. If the total curvature of Γ is less than 4π, then Γ bounds an embedded disc.

Applications to non-positively curved spaces

Theorem (Asymptotic Plateau problem, S.'17)

Let X be a CAT(0) space of rank 2. Let Γ be a rectifiable Jordan curve in the Tits boundary $\partial_T X$. Then there exists a Lipschitz map $u : \mathbb{R}^2 \to X$ which locally minimizes the area and whose image Q satisfies $\partial_T Q = \Gamma$.

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Theorem (Conformal change, L.-S.'17)

Let X be a CAT(0) space and f be a function on X, continuous, convex and bounded from below. Then the conformally equivalent space $e^f \cdot X$ is CAT(0).

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- Regularity of minimal surfaces in Finsler manifolds.
- Fine regularity of harmonic and minimal surfaces in CAT(0) spaces.

Thank you.

TOPOLOGICAL AND EQUIVARIANT RIGIDITY IN THE PRESENCE OF LOWER CURVATURE BOUNDS

SPP2026: Geometry at Infinity

Fernando Galaz-García (KIT) | Martin Kerin (WWU Münster)

November 9, 2017

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- Fang–Rong, Wilking,....
  $n \geq 8$, k sufficiently large $\implies$ strong restrictions on M^n.

Problem 1

Classify (M^6, g) *with* $\pi_1(M^6) = 0$, $\sec > 0$, *such that* $T^2 \curvearrowright (M^6, g)$.

GUIDING PROBLEMS

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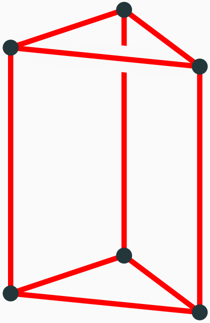
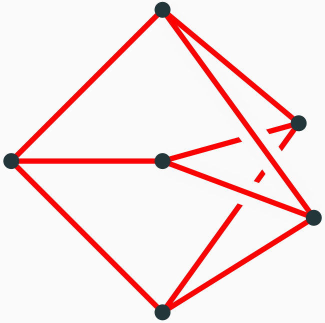


Graph for $\mathbb{C}P^3 \# \mathbb{C}P^3$



Graph for known examples with $\sec > 0$ and $\chi = 6$

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- More complicated than in manifold case:

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- Would need to classify all Alexandrov spaces with curv > 0:
  Y Alexandrov space with curv $> 0 \implies X = Y * \mathbb{S}^1$ fixed-point homogeneous Alexandrov space with curv > 0.

Spaces and moduli spaces of Riemannian metrics with curvature bounds on compact and noncompact manifolds

A. Dessai (Fribourg), B. Hanke (Augsburg), W. Tuschmann (KIT)

Further researchers: M. Amann (Augsburg), M. Bustamante (Augsburg), D. González-Álvaro (Fribourg), J. Wermerlinger (Fribourg)

SPP 2026: Kick off meeting, Potsdam
November 9-10, 2017

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Any subgroup G of the group $Diff(M)$ of diffeomorphisms of M acts on $\mathcal{R}^P(M)$. The moduli space of Riemannian metrics with property P with respect to the group G is

$$\mathcal{M}_G^P(M) = \mathcal{R}^P(M)/G$$

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distinguished classes of smooth M-bundles with structure group G and with a fiberwise geometric structure P

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By refining and extending these two methods to families of manifolds and then combining them with powerful tools in algebraic topology (e.g. smoothing and cobordism categories), researchers have succeeded in giving insight on the topology of $\mathcal{R}^{scal>0}(M)$ and their moduli, at least when M is *closed* simply connected spin manifold.

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For example:

- The non-multiplicativity (for fiber bundles) of the $\widehat{A}$-genus yields the detection of infinite order elements in $\pi_*\mathcal{R}^{scal>0}(M)$ which survive in homology under Hurewicz map [Hanke-Schick-Steimle].

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- There is a non-trivial map from $\pi_*\mathcal{R}^{scal>0}(M)$ to topological K-theory [Bottvinnik-Ebert-Randal Williams].

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- A more challenging goal is to construct bundles with non-simply connected and psc fiber whose total space has vanishing $\widehat{A}$-genus but some non-trivial *higher* $\widehat{A}$-genera. Succeeding here would lead to finding nontrivial families of psc metrics on closed manifolds with non-trivial fundamental group (and their universal covers).

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- Seven dimensional Witten spaces (Kreck-Stolz 1993, Kapovitch-Petrunin-Tuschmann 2005)

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- Seven dimensional Witten spaces (Kreck-Stolz 1993, Kapovitch-Petrunin-Tuschmann 2005)
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To distinguish components of the moduli space index-theoretical invariants due to Gromov-Lawson and Kreck-Stolz have been used, which require, in particular, that the manifolds in question are of dimension $4k-1 \geq 7$.

Moduli space for manifolds of dimension $\neq 4k-1$

Main goal: Exhibit examples of manifolds in dimension $\neq 4k-1$ for which the moduli space of metrics of nonnegative (or positive) sectional curvature has many connected components.

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In dimensions $\neq 4k-1$ reduced η-invariants of Dirac operators with coefficients in flat bundles can be used for non-simply connected manifolds.

η-invariants measure the spectral asymmetry and occur naturally in the Atiyah-Patodi-Singer index theorem for manifolds with boundary.

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This is work in progress ...

Fiber bundles with geometric structure and spaces of Riemannian metrics

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Some times this question can be tackled by understanding the orbit map of the action of $G \subset Diff(M)$ on $\mathcal{R}^P(M)$ and its orbit space $\mathcal{M}_G^P(M)$ and yields non-trivial classes in the higher homotopy groups of spaces of metrics or their moduli.

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- It is also natural to compare and study different topologies on spaces and moduli spaces of Riemannian metrics on both noncompact and compact manifolds. For example, for both compact and non-compact manifolds with non-negative sectional curvature, can one compare the homotopy types of the moduli space under the topologies of uniform C^k convergence $(0 \leq k \leq \infty)$? or what if one uses the Gromov-Hausdorff topology?

Diffeomorphisms and the topology of positive scalar curvature

Johannes Ebert, Thomas Schick, Wolfgang Steimle,
Thorsten Hertl, (cooperation: Diarmuid Crowley, Simone Cecchini,
Vito Zenobi)

Universität Münster, Universität Göttingen, Universität Augsburg
Universität Göttingen, University of Melbourne

Potsdam SPP meeting Nov 2017

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The diffeomorphism group $\mathrm{Diff}(M)$ acts on $\mathcal{R}^+(M)$ by pullback.
We are in particular studying the interplay between the homotopy theory of $\mathrm{Diff}(M)$ and of $\mathcal{R}^+(M)$. More specifically, we will analyze the induced map

$$\pi_k(\mathrm{Diff}(M)) \to \pi_k(\mathcal{R}^+(M)).$$

Positive scalar curvature specific tools

- Higher index theory of the Dirac operator, provided M has a spin structure,
- minimal hypersurfaces,
- specific to this project: constructions from homotopy theory and geometric topology.

Stabilization and Concordance for Diff(M)

Important tools in the study of $\pi_k(\mathrm{Diff}(M))$ are:

- stabilization: cartesian product with the identity on $[0,1]$: $\pi_k(\mathrm{Diff}(M)) \to \pi_k(\mathrm{Diff}(M \times [0,1]))$,
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- stability results (in a range of k the stabilization map induces an isomorphism),
- concordance: "loosen the relation from *homotopy* to *concordance*:
  - compare $\pi_1(\mathrm{Diff}(M))$ with $(\pi_0(\mathrm{Diff}(M \times [0,1]))$ rel boundary).
  - Systematically done by the introduction of concordance spaces.
  - This involves simplicial methods, which are employed systematically.

Stability and concordance for $\mathcal{R}^+(M)$

The PhD project of Thorsten Hertl is concerned with

- constructing a concordance space $CS(M)$ and a (realization of a) concordance category $BCC(M)$ of metrics of positive scalar curvature on M,
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- establish how close the comparison maps are to being homotopy equivalences. The spaces might be analyzed with the help of *cobordism categories*,
- getting examples where there are differences. To achieve this, we plan to exploit the minimal hypersurface method of Schoen and Yau.

More about diffeos and psc

- Factorize the action of $\mathrm{Diff}(M)$ on $\mathcal{R}^+(M)$ through suitable cobordism categories. Determine the kernel, which acts rigidly.
- Investigate the rich additional index theoretic tools which appear if $\pi_1(M)$ is non-trivial: conjecturally $\pi_*(\mathcal{R}^+(M))$ is then at least as large as $KO_*(C^*_{red}\pi_1(M))$, but much of this remains very mysterious.

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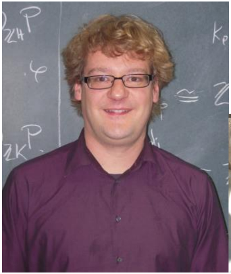
- Effects of stabilization for $\mathcal{R}^+(M)$.
- Explicit geometric constructions of elements of $\pi_k(\mathrm{Diff}(M))$ and the resulting elements in $\pi_k(\mathcal{R}^+(M))$ (instead of existence results via homotopy theory).
- Study of complete metrics of psc on non-compact M: what is the effect of the many local constructions we know in this context?

Further members of the team

Because of childcare, distance, and a previous commitment to another conference some members of our team can't be present:




Johannes Ebert, Diarmuid Crowley, Wolfgang Steimle, Vito Zenobi

Gerbes in Renormalization and Quantization of Infinite-Dimensional Moduli Spaces

M. Upmeier

Mathematical Institute
University of Oxford

November 2017

What are gerbes?

Basics

Definition

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- Complex line bundles represent classes in $H^2(X;\mathbb{Z})$ via c_1.
- Which cohomology? Let A_X be a sheaf on X, such as

$$\mathcal{O}_X, C^\infty_X, \underline{\mathbb{Z}}_X, \mathbb{Z}(p)_X$$

and take sheaf cohomology $H^*(A_X)$.

What are gerbes?

Concrete models for $H^3(X; \mathbb{Z})$

- (Hitchin–Chatterjee) Pick open cover $\{U_i\}$ of X and make explicit the cocycle in sheaf cohomology:
  1. Line bundle L_{ij} on $U_i \cap U_j$.
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- (Murray) A <u>bundle gerbe</u> consists of a submersion

$$(\sqcup U_i =)\, Y \xrightarrow{\pi} X,$$

a line bundle L on $Y \times_X Y = (\sqcup U_i \cap U_j)$, and

$$m\colon\ \mathrm{pr}_{12}^* L \otimes \mathrm{pr}_{23}^* L \xrightarrow{\cong} \mathrm{pr}_{13}^* L.$$

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- (Brylinski) Bundles of groupoids on X.
- (Dixmier) Bundles of projective Hilbert spaces on X: $BPU(\mathcal{H}) \sim K(3,\mathbb{Z})$ for Hilbert space $\mathcal{H} \cong \ell^2$.

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Example

Σ=Riemann surface, X=complex manifold. $\mathcal{M}$=moduli space of J-holomorphic curves.

$$\begin{tikzcd}
\Sigma \times \mathcal{O}(\Sigma, X) \arrow[r, "\mathrm{eval}"] \arrow[d, "\mathrm{pr}_2"'] & X \\
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Take $c \in H^4(X; \mathbb{Z})$, pull back, and fiber integrate along Σ.
Get class in $H^2(\mathcal{O}(\Sigma, X); \mathbb{Z})$, the first Chern class of L.

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- There is a gerbe on X that represents $L_{\bar\partial}$. It can be constructed from the eigenvalues and eigenspaces of chart transition functions for X.
- If $b_3(X) = b_4(X) = 0$, $\dim X = 6$, then $L_{\bar\partial}$ is trivial.

Some interesting problems

- <u>Yang–Mills theory</u>: show that the determinant line bundle has a flat connection (open in general. Over the regular stratum: Hitchin).

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- Orientations on the moduli space of G_2-instantons (joint with D. Joyce) or the moduli space of associative 3-folds.

Analytic L^2-invariants of non-positively curved spaces

Holger Kammeyer, Steffen Kionke, Roman Sauer, Thomas Schick

KIT, Karlsruhe
University of Göttingen

November 09, 2017

Analytic L^2-invariants

Let M be a finite volume Riemannian manifold with universal covering $\widetilde{M}$. We consider the Laplacian Δ_p acting on square-integrable p-forms on $\widetilde{M}$:

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- The p-th **Novikov–Shubin invariant** $\alpha_{p,an}(\widetilde{M})$ detects the amount of spectral mass the operator Δ_p has near zero.

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$$\Delta_p \colon L^2\Omega^p(\widetilde{M}) \to L^2\Omega^p(\widetilde{M}).$$

Analytic L^2-invariants capture spectral properties of Δ_p.

- The p-th **$\mathbf{L^2}$-Betti number** $b^{(2)}_{p,an}(\widetilde{M})$ quantifies the size of the space $\ker \Delta_p$ of L^2-harmonic p-forms.
- The p-th **Novikov–Shubin invariant** $\alpha_{p,an}(\widetilde{M})$ detects the amount of spectral mass the operator Δ_p has near zero.
- The **$\mathbf{L^2}$-torsion** $\rho^{(2)}_{an}(\widetilde{M})$ is the L^2-counterpart to analytic Ray–Singer torsion of M.

Locally symmetric spaces

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- Show $\alpha_p(\widetilde{M}) < \infty$ for some p whenever $b_q^{(2)}(\widetilde{M}) = 0$ for all q.

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- Compute $\rho^{(2)}(\widetilde{M})$ if M is odd-dimensional.

  ME proportionality of $\mathbf{L^2}$-torsion *as conjectured by Lück, Sauer and Wegner predicts non-zero values for* $G = \mathrm{SL}_3(\mathbb{R})$ *and* $\mathrm{SL}_4(\mathbb{R})$.

Finite volume Kähler hyperbolic manifolds

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Hence M, albeit non-compact, satisfies the following conjecture.

Conjecture (Chern–Hopf)

Let M be a closed, negatively curved $2k$-manifold. Then $(-1)^k \chi(M) > 0$.

Further directions

Other parts of the project include:

- An L^2-index theorem for manifolds with cusp ends.
- Equivalence of twisted combinatorial and analytic L^2-torsion.
- L^2-torsion as a functional on representation varieties.

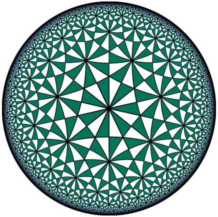
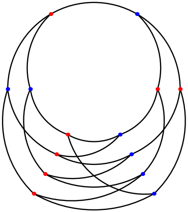
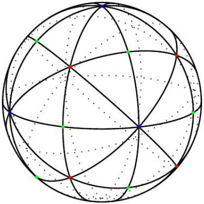
Parabolics and Invariants

Benjamin Brück, Kai-Uwe Bux, Dawid Kielak,
Eduard Schesler, Stefan Witzel

Potsdam

9.11.2017

Parabolic subgroups occur as stabilisers.

Lie groups	Algebraic groups	Coxeter groups
		
Symmetric spaces	Buildings	Coxeter complexes

Example: Parabolics in $\mathrm{GL}_n(R)$

$\mathrm{GL}_n(R) = \{\text{invertible } n \times n \text{ matrices over } R\}$

$$\begin{pmatrix} * & * & * & * \\ * & * & * & * \\ * & * & * & * \\ * & * & * & * \end{pmatrix}$$

Parabolics: given block diagonal structure

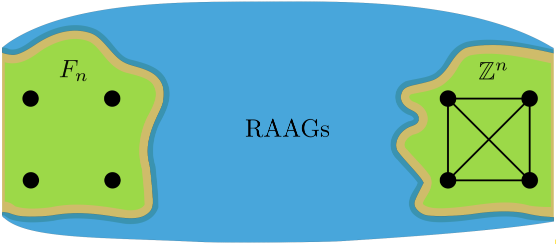
$$\begin{pmatrix} * & * & * & * \\ * & * & * & * \\ & & * & * \\ & & * & * \end{pmatrix}$$

Borel subgroup $\mathbf{B}(R)$ (a minimal parabolic)

$$\begin{pmatrix} * & * & * & * \\ & * & * & * \\ & & * & * \\ & & & * \end{pmatrix}$$

PhD project of Benjamin Brück (Project I)

- Describe the free factor complex in terms of parabolic subgroups in $\mathrm{Aut}(F_n)$.
- Find an appropriate notion of parabolic subgroups in $\mathrm{Aut}(\mathrm{RAAG})$ in order to obtain a complex interpolating between the free factor complex and the building associated to $\mathrm{GL}_n(\mathbb{Q})$.

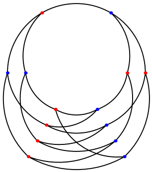


Σ-invariants

There is a sequence of invariants $\Sigma^n(G) \subset Hom(G, \mathbb{R})$, which record which homomorphism $\phi\colon G \to \mathbb{R}$ has kernel with good topological properties (a small classifying space – type F_n).

PhD project of Eduard Schesler (Project II)

- Prove that the Σ-invariants of a Borel subgroup $G = \mathbf{B}(\mathbb{Z}[1/p])$ are determined by a polyhedron $P \subset Hom(G, \mathbb{R})$ in the sense that $\Sigma^{k+1}(G)$ is the complement of the k-skeleton $P^{(k)}$.
- Determine the connectivity of subcomplexes of buildings arising as preimages of retractions.



Project III – Σ- and L^2-invariants

$\Sigma^1(G)$ can be computed using L^2-invariants for

- 3-manifold groups,
- 2-generator 1-relator groups (knot complement groups).

We would like to do the same for free-by-cyclic groups $F_n \rtimes \mathbb{Z}$, more general $\mathrm{RAAG} \rtimes \mathbb{Z}$, and the Borel subgroup $\mathbf{B}(\mathbb{Z}[1/p])$.

Open problem

Are there finitely many connected components of $\Sigma^1(F_n \rtimes \mathbb{Z})$? (This is true for 3-manifolds.)

THANK YOU!

Compactifications and Local-to-global Structure for Bruhat-Tits Buildings

Linus Kramer (Münster) and
Petra Schwer (Karlsruhe)

Kick-Off SPP 2026
Potsdam, November 2017

Main theme and goals

Recover global structures from restricted or sparse data.

Rigidity asks to identify isomorphic spaces in one class given an isomorphism between related objects in another class.
A crucial tool for proving rigidity are compactifications.

We want to study compactifications of Bruhat-Tits buildings.

Local-to-global principles ask to identify geometric structures from local data.
Roughly speaking a metric space X is *local-to-global rigid* if every space Y which locally looks like X is covered by X.

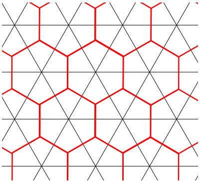
We want to study LG-rigidity of Bruhat-Tits buildings.

Local-to-global rigidity

De La Salle and Tessera discovered that the 1-skeleton of the Bruhat–Tits building for the group $\mathrm{SL}_n(K)$ is *not* LG-rigid if $K = \mathbb{F}_p((t))$ but is LG-rigid if $K = \mathbb{Q}_p$.
Indeed, these buildings look alike on a small scale but on an intermediate scale their metric balls look different.

Main objectives:

1. Study LG-rigidity of the **chamber graph**.
2. Explain geometrically the **failure of LG-rigidity** in the function field case.
3. Study LG-rigidity for **other types** than $\tilde{A}_n$.



1-skeleton and chamber graph (red) in type $\tilde{A}_2$.

Compactifications

We consider the horofunction compactification of Euclidean buildings X equipped with an *asymetric metric* δ.

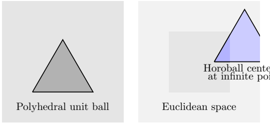
This generalizes e.g. Satake compactifications, and avoids both Bruhat–Tits Theory of reductive groups and the machinery of Berkovich spaces.

Main objectives:

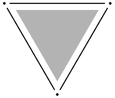
1 Give a **uniform construction** of compactifications.

2 Study their **functorial properties**.

3 Study the **dynamics** of discrete group actions on buildings.



Euclidean space   Nonstandard euclidean space



The stratified space of horofunctions

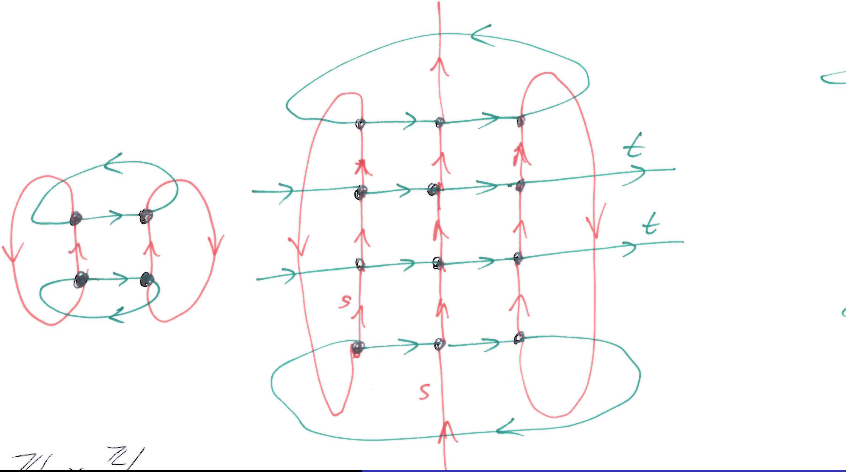
Asymptotic geometry of sofic groups and manifolds

Vadim Alekseev, Andreas Thom

Institut für Geometrie, TU Dresden

SPP 2026 Kick-Off Meeting, Potsdam

Consider the following approximation of $\mathbb{Z}^2 = \langle s, t \rangle$:



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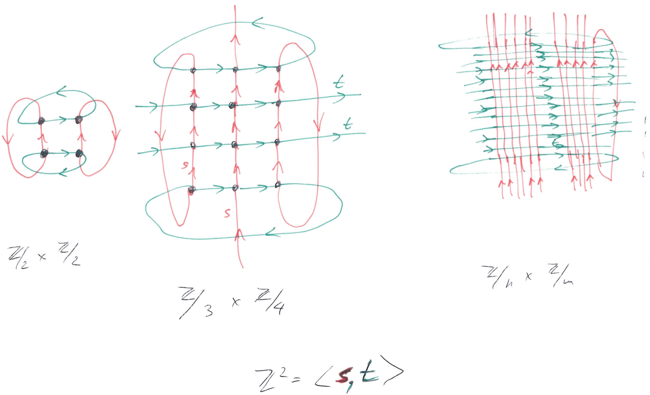
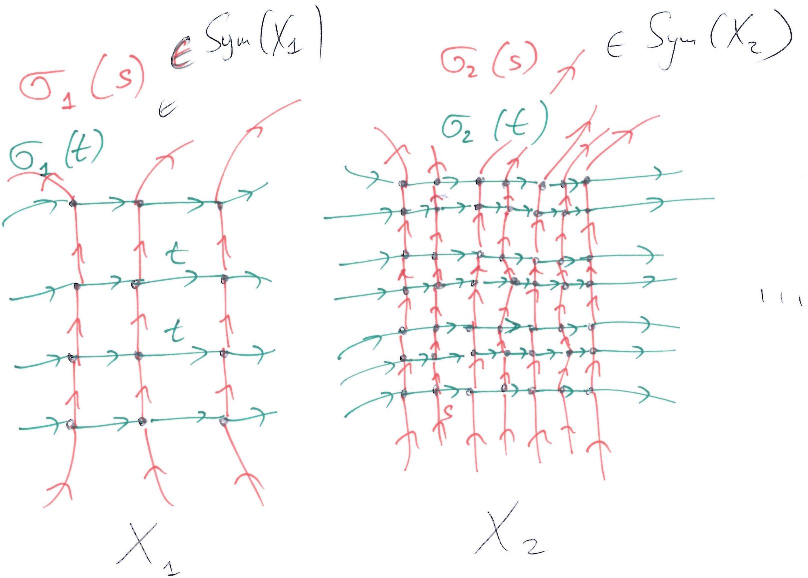


...and observe that we can alter it at the “edges” keeping s, t to be permutations:


$\sigma_1(s)$
$\in \mathrm{Sym}(X_1)$
$\in$
$\sigma_1(t)$
t
t
X_1
$\sigma_2(s)$
$\in \mathrm{Sym}(X_2)$
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s
X_2
...

If $\Gamma = \langle s, t, r, \ldots \rangle$ is a group, a sofic approximation of Γ is a sequence of graphs

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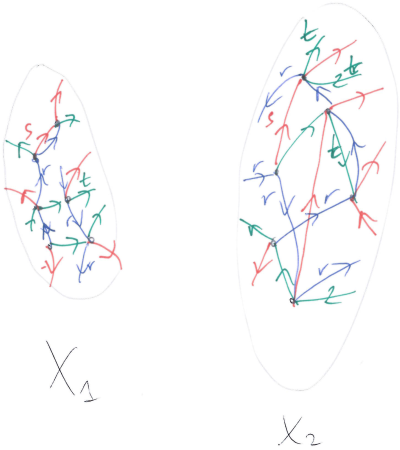
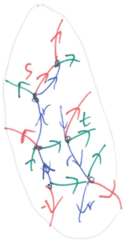
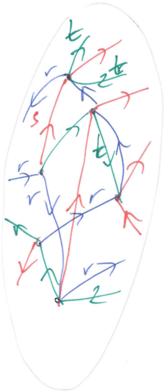


(|)

If $\Gamma = \langle {\color{red}s}, {\color{teal}t}, {\color{blue}r}, \ldots \rangle$ is a group, a sofic approximation of Γ is a sequence of graphs "looking more and more like Γ almost everywhere":



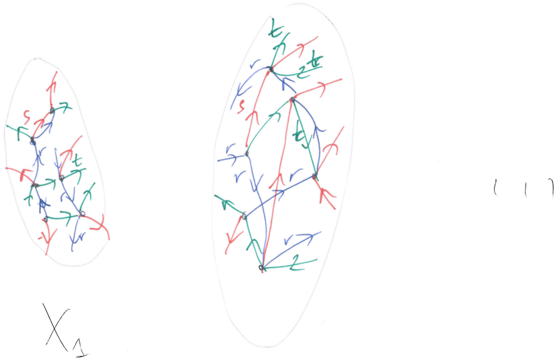
X_1



X_2

Definition

If Γ is a group, a sofic approximation of Γ is a sequence of graphs labelled by generators of Γ converging to the Cayley graph of Γ in the sense of Benjamini–Schramm.



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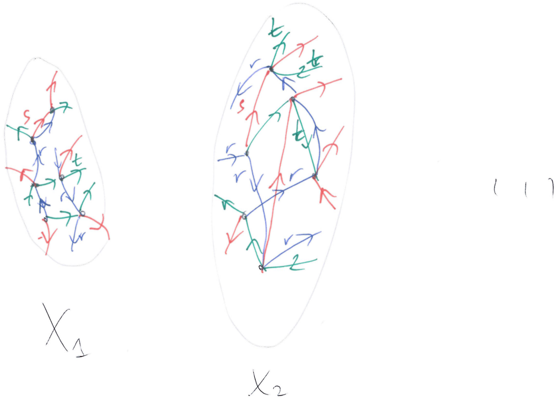
A group Γ possessing a sofic approximation is called sofic.



Definition

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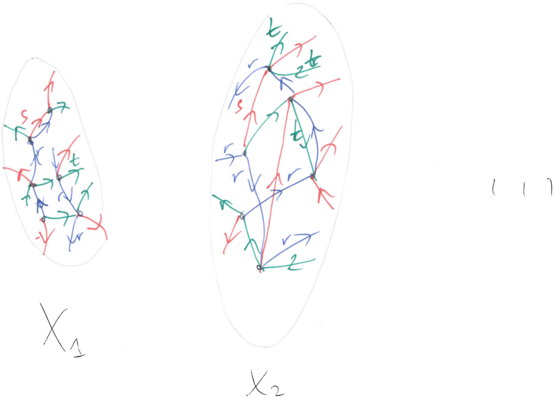
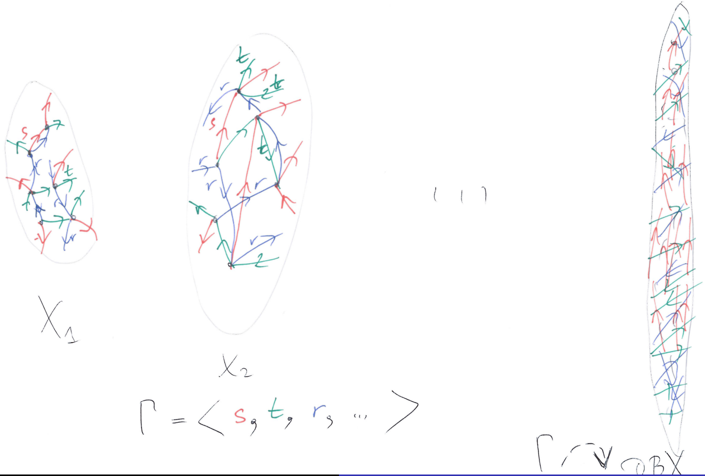
Major open question: is every group sofic?



Given a sofic approximation $\sigma_i \colon \Gamma \to \mathrm{Sym}(X_i)$, we can look at its "geometry at infinity", here represented by the *Stone-Čech boundary* $\partial\beta X$ of it:



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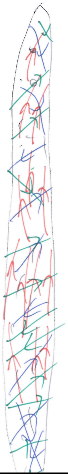

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Question

When is an action a sofic boundary action? Can one characterise soficity of a group by the existence of special action on a topological or measure space?

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Question

How do dynamical invariants of the boundary action relate the group and the geometry of its sofic approximation?

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Theorem (A.– Finn-Sell, '16)

Let Γ be a sofic group, $\mathcal{X}$ a sofic approximation of Γ, and X be the space of graphs constructed from $\mathcal{X}$. Then geometric properties of X – such as property A, coarse embeddability into Hilbert space, geometric property (T) – imply the corresponding approximation properties of Γ: amenability, a-T-menability resp. Kazhdan's property (T).

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Question

To which extent do the converse statements hold?

One can combine the aforementioned ideas to study *higher-dimensional* sofic approximations.

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Example

If Γ is sofic and the classifying space $B\Gamma$ is compact, then $E\Gamma$ is a limit of finite-dimensional cell complexes in the Benjamini-Schramm sense.

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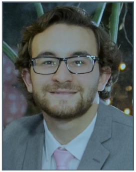
Question

Do natural sequences of invariants, such as mean Pontryagin numbers or mean Chern numbers exist along a sofic approximation?

Vadim Alekseev



Leonardo Biz
(starting December 2017)

Rahel Brugger
(starting April 2018)

Andreas Thom

Invariants and boundaries of spaces

Andreas Ott, Heidelberg

I. Bounded cohomology via partial differential equations

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II. Simplicial volume and polylogarithms

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III. Asymptotics of Higgs bundles, harmonic maps, and pleated surfaces

Joint work with Tobias Hartnick (Technion).

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Bounded cohomology is notoriously difficult to compute, only few cases are known.

But it is expected to be more accessible for Lie groups G.

Conjecture (Dupont, Monod): Let G be a connected semisimple Lie group with finite center. Then

$$H^n_{\mathrm{cb}}(G;\mathbb{R}) \cong H^n_{\mathrm{c}}(G;\mathbb{R}) \quad \text{for all } n \geq 0.$$

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The PDE technique also applies to lattices in Lie groups by passing to more general coefficients, e.g. we aim to compute the bounded cohomology of the free group $\mathbb{F}_2$.

Joint work with Michelle Bucher (Geneva) and Andreas Potschka (IWR Heidelberg).

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The simplicial volume of closed locally symmetric spaces of the noncompact type is known to be positive, and has been computed explicitly for closed real-hyperbolic manifolds and those locally covered by $\mathbb{H}^2 \times \mathbb{H}^2$.

Goal: Compute the simplicial volume of closed manifolds locally covered by $SL_3(\mathbb{R})/SO(3)$ in terms of the trilogarithm function.

This amounts to computing the Gromov norm of the volume class in the continuous bounded cohomology $H^5_{\mathrm{cb}}(SL_3(\mathbb{R}); \mathbb{R})$.

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Joint work with Michelle Bucher (Geneva), Tobias Hartnick (Technion), Guido Kings (Regensburg), and Clara Löh (Regensburg).

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DDW only consider Φ, while MSWW keep track of both A and Φ.

Goal: If the Φ_∞ in DDW correspond to trees, determine what geometric objects in $\mathbb{H}^3$ the limiting pairs (A_∞, Φ_∞) in MSWW correspond to.

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Answer in progress: They correspond to pleated surfaces in $\mathbb{H}^3$, obtained by high-energy degeneration of the harmonic maps $u : \widetilde{\Sigma} \to \mathbb{H}^3$, with total bending angle determined by the holonomy of the limiting connection A_∞.

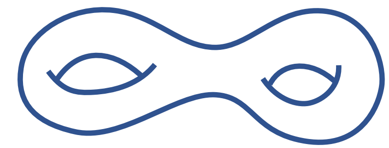
Hitchin components for orbifolds

Daniele Alessandrini (Heidelberg)
Gye-Seon Lee (Heidelberg)

9 November 2017

Character Varieties

S = closed oriented surface of genus $g \geq 2$



$G = PGL(n, \mathbb{R})$

$$X(\pi_1(S), G) = \mathrm{Hom}^*\,(\pi_1(S), G)/G$$

Important in

Higher Teichmüller Theory

Knot Theory

Integrable Systems

Gauge Theory

Geometric Quantization

SUSY Quantum Field Theories

& other areas of Geometry and Theoretical Physics

(Higher) Teichmüller theory

$\mathcal{T}(S) =$ {Fuchsian representations} $\subset X(\pi_1(S), PGL(2, \mathbb{R}))$

$\mathcal{T}(S)$ is a connected component of $X(\pi_1(S), PGL(2, \mathbb{R}))$.

Generalize to $PGL(n, \mathbb{R})$

$\boldsymbol{Hit(S, n)} \subset X(\pi_1(S), PGL(n, \mathbb{R}))$

The **Hitchin component** is a connected component of X.

It is homeomorphic to a ball of dimension $2(n^2 - 1)(g - 1)$.

It shares many properties with $\mathcal{T}(S)$.

It is a higher Teichmüller space.

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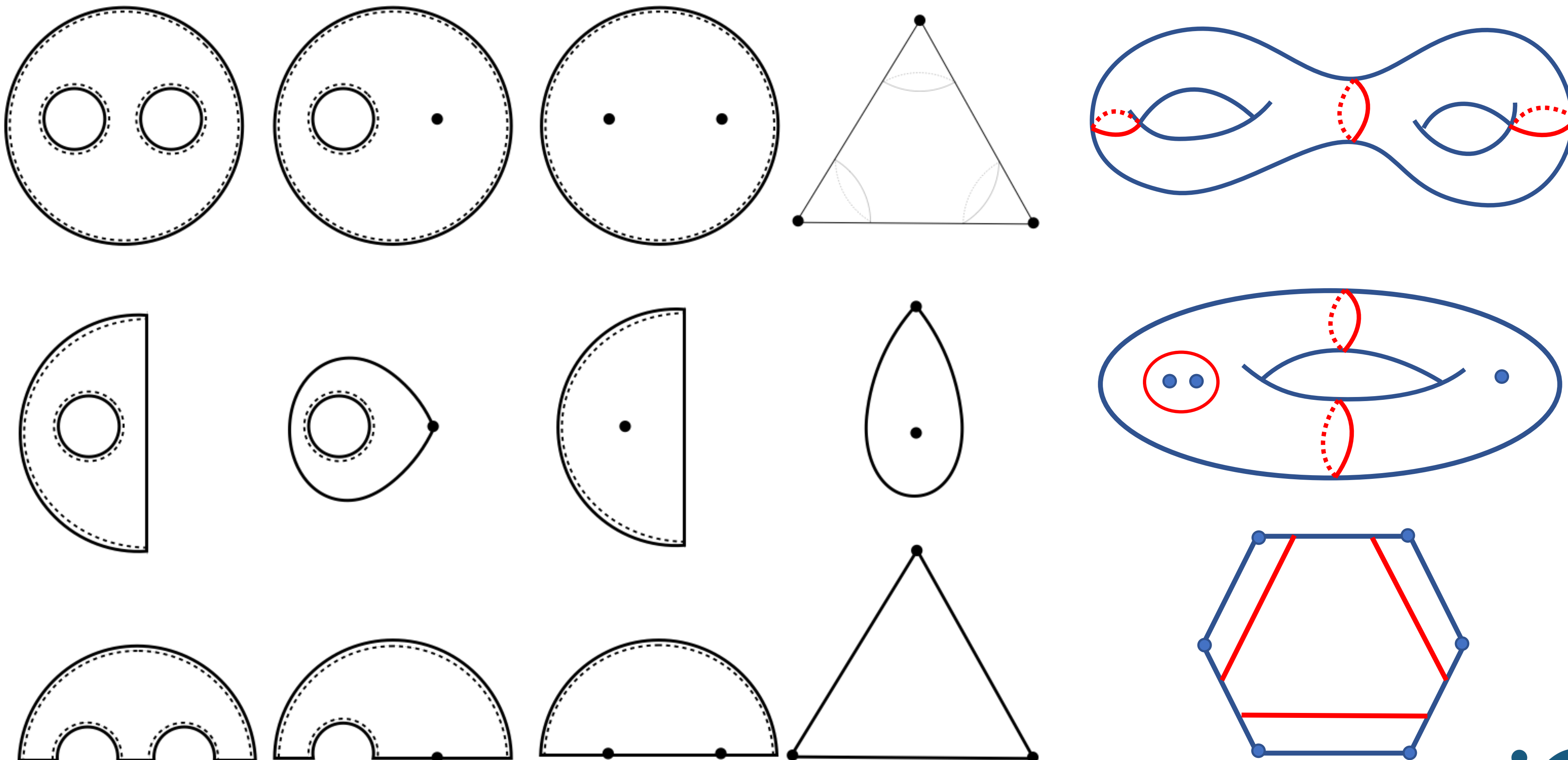
2-dimensional orbifold



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Topology of Hitchin componts of orbifolds

Theorem (A-L-Schaffhauser, for n=2, 3, Thurston, Choi-Goldman)

Let Σ be a compact 2-orbifold with $\chi(\Sigma) < 0$.

If Σ is orientable and of genus g with k cone points of order $m_1, \ldots, m_k$, then the Hitchin component of $\pi_1(\Sigma)$ in $PGL(n, \mathbb{R})$ is homeomorphic to a ball of dimension

$$2(n^2-1)(g-1) + 2\sum_{i=1}^{k}\sum_{d=2}^{n}\left\lfloor d\left(1-\frac{1}{m_i}\right)\right\rfloor$$

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Applications

- We can classify all the rigid orbifolds, where the Hitchin component has dimension 0.
- There are some orbifolds with very "small" Hitchin component, of dimension 1, 2. Their geometry can probably be understood completely.
- The Hitchin components of orbifolds are submanifolds of the Hitchin components of surfaces, with very high symmetry (totally geodesic).
- Describe parameter spaces of $\mathbb{R}P^3$-structures on some Seifert fibered 3-manifolds. We get many examples of closed 3-manifolds with a rigid $\mathbb{R}P^3$-structure.

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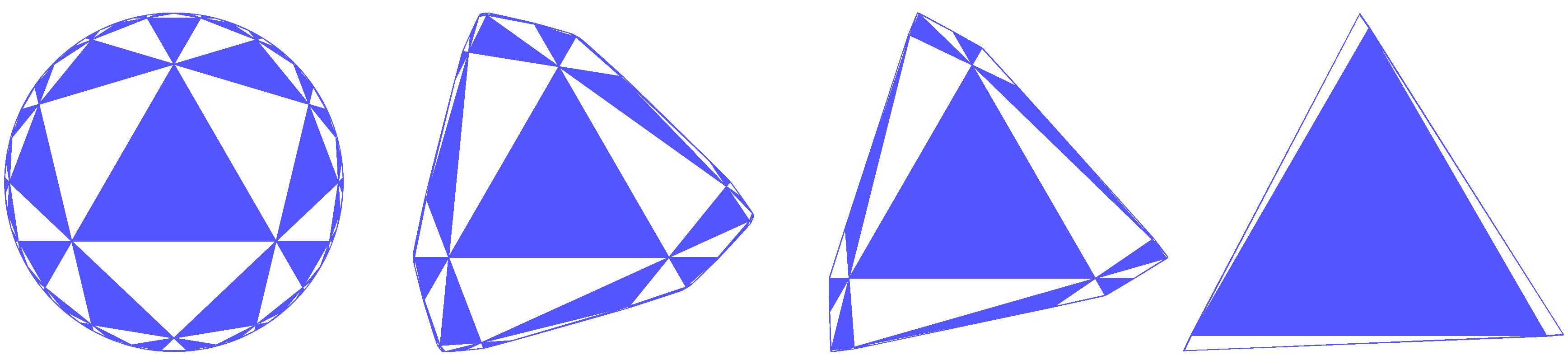
Project 1: Degeneration of Hitchin representations for small orbifolds

Behavior at infinity of Hitchin components: What happens when a sequence of representations diverges?

Understand this problem for Hitchin components of dimension 1.



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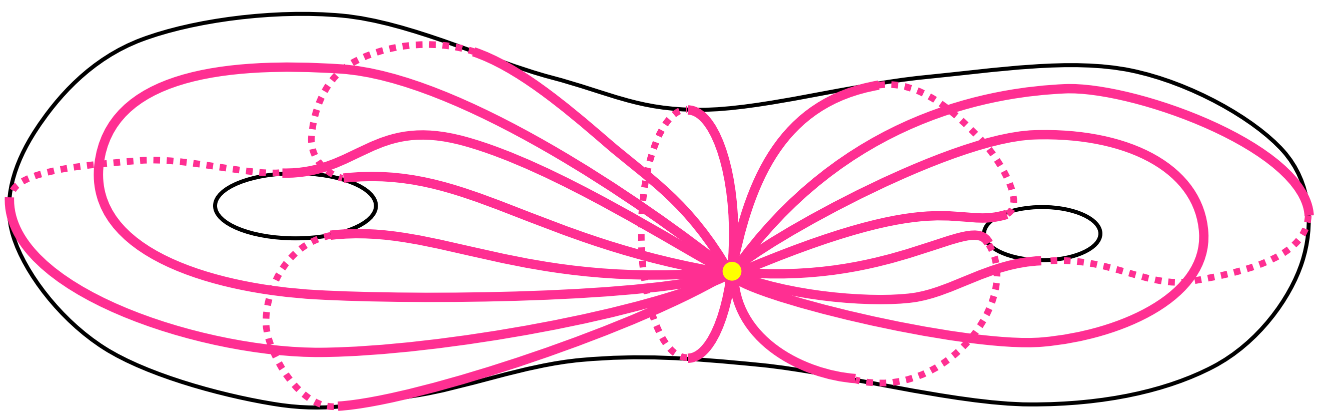
Project 2: Parametrization of Hitchin components for orbifolds

As the Fenchel-Nielsen coordinates played an important role for the Teichmüller theory, it will be useful to find a geometric parametrization of orbifold Hitchin components.



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Anosov reprentations and Margulis spacetimes

Sourav Ghosh

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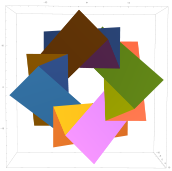
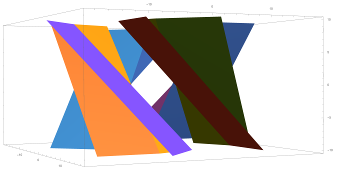
November 7, 2017

Objects of study

1. Anosov representations of word hyperbolic groups into semisimple Lie groups:
   - the Gromov boundary of the hyperbolic group embeds into some appropriate flag variety,
   - admits some appropriate Anosov dynamics.
2. Margulis spacetimes:
   - quotient manifolds of affine spaces under proper affine actions of word hyperbolic groups.





Known results

- Labourie, Guichard–Wienhard: Anosov representations are stable.
- Bridgeman–Canary–Labourie–Sambarino: moduli spaces of Anosov representations admit pressure metrics.
- Margulis, Drumm, Abels–Margulis–Soifer, Smilga: existence and examples of Margulis spacetimes.
- Goldman–Labourie–Margulis, Ghosh, Ghosh–Treib: necessary and sufficient conditions for proper actions of word hyperbolic subgroups of $\mathrm{SO}^0(n+1,n) \ltimes \mathbb{R}^{2n+1}$ with Anosov linear parts.
- Ghosh: Moduli spaces of aforementioned Margulis spacetimes admit pressure metrics with nice properties.

Main goals

- Provide necessary and sufficient conditions for proper actions of word hyperbolic subgroups of general affine Lie groups with Anosov linear parts.
- Define and study properties of the pressure metric on the moduli space of general Margulis spacetimes.
- Investigate the boundaries of the moduli space of Anosov represenations.

Proposed methods

- Define continuous versions of the normalised vector valued Margulis invariants introduced by Smilga and use them to give necessary and sufficient conditions for proper actions.
- Use Anosov structure to endow the moduli space of Margulis spacetimes with pressure metric.
- Use affine cross ratios to study the pressure metric.
- Use Margulis spacetimes admitting parabolics to study boundaries of the moduli space of Anosov representations.

Thank You!

Rigidity, deformations and limits of maximal representations

Federica Fanoni Maria Beatrice Pozzetti Anna Wienhard

Heidelberg University

Maximal representations

S closed surface of genus $g \geq 2$

Maximal representations

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Maximal representations: special representations

$$\rho : \pi_1(S) \to \mathrm{Sp}(2n, \mathbb{R})$$

or more generally into G Hermitian Lie group

$$\rightsquigarrow \mathrm{Rep}_{\mathrm{max}}(\pi_1(S), G) \subset \mathrm{Hom}(\pi_1(S), G)/G$$

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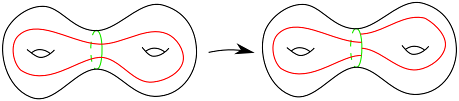
"Nice":

- Discrete and faithful $\rightsquigarrow$ get discrete subgroups of G
- Generalizing *Teichmüller space* $\mathrm{Teich}(S)$ (space of marked hyperbolic structures on S)
- $\mathrm{Rep}_{\mathrm{max}}(\pi_1(S), G)$ is a union of connected components $\rightsquigarrow$ deformations and points at infinity

Start with a hyperbolic surface $X \in \mathrm{Teich}(S)$.
Basic move: left *twist* about a simple closed geodesic



Earthquakes: deformations of hyperbolic surfaces

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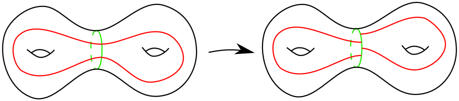

Thurston's idea: left twist about a *geodesic lamination* (closed subset of S foliated by complete geodesics) $\rightsquigarrow$ left *earthquakes*

Theorem (Thurston)

Any two points in Teichmüller space can be joined by a left earthquake.

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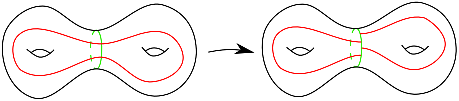

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By varying the amount of twisting, we get a flow in $\mathrm{Teich}(S)$, called *earthquake flow*.

Earthquakes: deformations of maximal representations?

Goal: define earthquakes for maximal representations

Some questions:

- Which representations can be joined by an earthquake?

Note: $\mathrm{Teich}(S)$ is connected, while $\mathrm{Rep}_{\mathrm{max}}(\pi_1(S), \mathrm{Sp}(2n, \mathbb{R}))$ isn't.

- Can we define an earthquake flow? What are its properties?

Projektive Flächen, Segre Strukturen und die PSL(n,R)-Hitchin Komponente

Thomas Mettler

GOETHE UNIVERSITÄT FRANKFURT AM MAIN

SPP 2026 Kickoff Meeting
Potsdam, November 9-10, 2017

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Paths on a manifold

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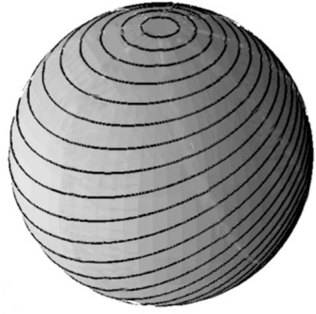

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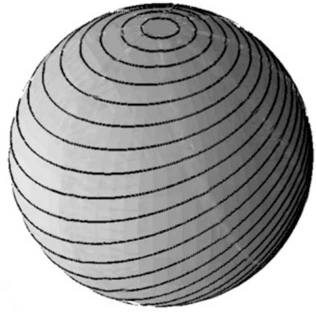

Prescription $\mathfrak{p}$ of a path γ for every $p \in M$ and every direction $[v] \in T_p M$.

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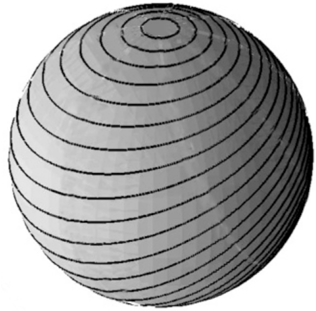

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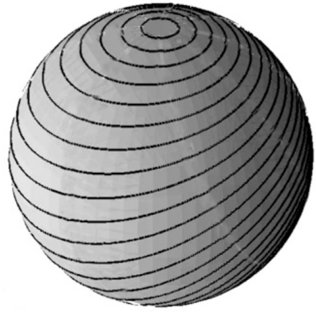

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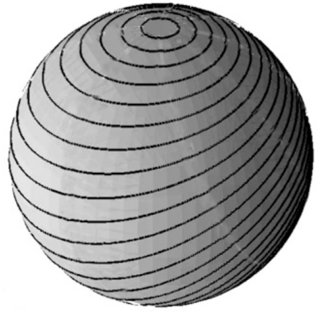

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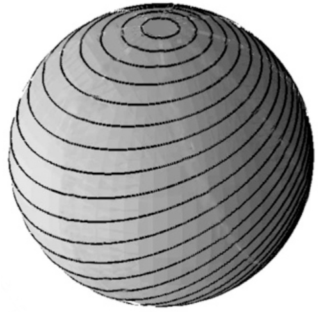
Characterisation due to Bryant, Dunajski & Eastwood.

Paths on a manifold



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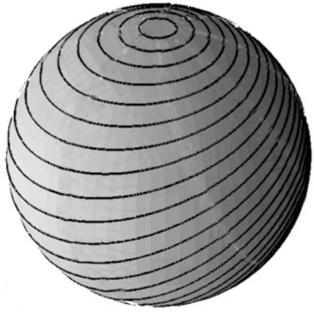

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Gives a functional on the space of Riemannian metrics.

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> Choice of $[g]$ on $(M^2, \mathfrak{p})$ gives **principal bundle connection** A and “**Higgs field**” Φ s.t. variational equations are
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> Strongly reminiscent of Hitchin's Higgs bundle equations.

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Problem. Existence of critical points?

Problem. Uniqueness of critical points?

Problem. Relation of $\mathcal{F}_\mathfrak{p}$ to the energy functional on Teichmüller space introduced by Donaldson/Labourie?

Asymptotic geometry of the Higgs bundle moduli space

Jan Swoboda (LMU München) & Hartmut Weiss (CAU Kiel)

Potsdam, November 9-10, 2017

SPP 2026
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Asymptotic geometry of the Higgs bundle moduli space

SPP funded coworkers:

- Claudio Meneses (postdoctoral fellow)
- Seven Marquardt (PhD student)

Scientific goals:

Investigation of moduli spaces of solutions of Higgs bundles and related objects *in the large*:

- degenerations of solutions to the self-duality equations defining the moduli spaces
- asymptotic profile of underlying geometric structures such as their hyperkähler metrics
- identification of the resulting *geometry at infinity*

Asymptotic geometry of the Higgs bundle moduli space

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Higgs bundles

Hitchin's self-duality equation

X compact Riemann surface, $\pi\colon E \to X$ complex vector bundle,
$h\colon E \times E \to \mathbb{C}$ a hermitian metric on E.

Structures

- $\bar{\partial}_E : \Gamma(E) \to \Omega^{0,1}(E)$ holomorphic structure
- $d_A : \Gamma(E) \to \Omega^1(E)$ unitary connection

Hitchin's equation

$$\begin{aligned} F_A^{\perp} + [\Phi \wedge \Phi^*] &= 0 \\ \bar{\partial}_A \Phi &= 0 \end{aligned}$$

where $F_A^{\perp}$ is the trace-free part of the curvature.

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Motivation

Nonabelian Hodge theory & Kobayashi-Hitchin correspondence

For simplicity: $\deg(E) = 0$.

Theorem (Hitchin, Simpson)

If $(\bar{\partial}_{A_0}, \Phi_0)$ is a stable pair, then there exists a solution to Hitchin's equation (A, Φ) in the $\mathcal{G}_{\mathbb{C}}$-orbit of (A_0, Φ_0). (A, Φ) is irreducible and unique up to the action of $\mathcal{G}$.

Theorem (Donaldson, Corlette)

If $\deg(E) = 0$ then there is a $1:1$ correspondence

$$\{\text{irreducible solution of Hitchin's equation } (A, \Phi)\}/\mathcal{G}$$
$$\longleftrightarrow \{\text{irreducible representations } \rho \colon \pi_1(X) \to \mathrm{GL}(r, \mathbb{C})\}/\mathrm{GL}(r, \mathbb{C})$$

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Consider sequence (A_n, Φ_n) of solutions:

- $\|\Phi_n\|_{L^2} \leq C < \infty$: Uhlenbeck compactness $\Longrightarrow$ (A_n, Φ_n) subconverges to solution (A_∞, Φ_∞)
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Results

(Partial) compactification by limiting configurations

Theorem (MSWW)

For each limiting configuration (A_∞, Φ_∞) *there exists a family* (A_t, Φ_t) *of solutions to Hitchin's equation*

$$F_{A_t} + t^2[\Phi_t \wedge \Phi_t^*] = 0, \quad \bar{\partial}_{A_t}\Phi_t = 0$$

such that $(A_t, \Phi_t) \to (A_\infty, \Phi_\infty)$ *as* $t \to \infty$ *locally uniformly on* $X^\times$ *and exponentially fast in* t.

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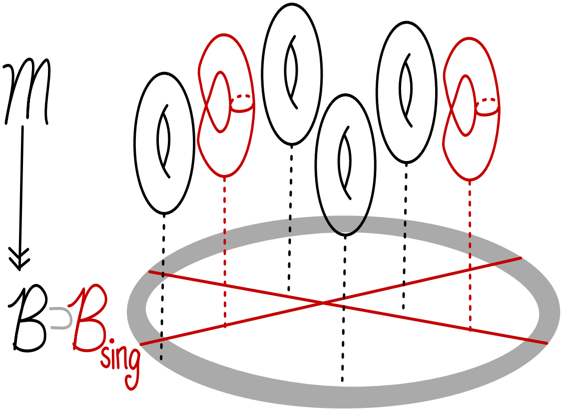

The semiflat metric

The GMN conjecture

Gaiotto, Moore and Neitzke describe (on a conjectural level) a procedure how to correct the semiflat metric g_{sf} on $\mathcal{M}$ (associated with the integrable system data) in order to obtain a complete HK-metric on $\mathcal{M}$.

These corrections are exponentially small in t and conjecturally give back Hitchin's L^2-metric.

Conjecture (Gaiotto, Moore, Neitzke)

$\exists\ \lambda > 0$ s.t.

$$g_{L^2} = g_{sf} + O(e^{-\lambda t}).$$

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Research goals within the SPP project

- Asymptotic geometry of branes and Hitchin fibers (Sven Marquardt)
- Kähler and analytic structures in moduli spaces originating from families of field theories (Claudio Meneses)
- Behavior of $\mathcal{M}$ under degenerations of the Riemann surface (Jan Swoboda)
- Exponential versus polynomial asymptotics (with Andy Neitzke, Rafe Mazzeo)
- Parabolic Higgs bundle moduli space on 4-punctured sphere as an ALG gravitational instanton (with Laura Fredrickson, Rafe Mazzeo)
- Gluing theorem for higher order zeroes (with Laura Fredrickson, Rafe Mazzeo)
- Harmonic maps and pleated surfaces (Jan Swoboda with Andreas Ott, Richard Wentworth and Mike Wolf)
- Asymptotic geometry of hyperpolygon spaces (Hartmut Weiss with Steven Rayan)

Asymptotic geometry of moduli spaces of curves

Roger Bielawski & Carolin Peternell
Leibniz Universität Hannover

Potsdam, November 9, 2017

Twistor spaces

- Several important geometric structures can be constructed and studied via their *twistor space*, i.e. as a parameter space of (real) rational curves in a complex manifold.
- The main example are hyperkähler and hypercomplex manifolds. The twistor space of such a manifold M^{4n} is a complex $(2n+1)$-dimensional Z manifold diffeomorphic to $M \times S^2$, fibring over $\mathbb{CP}^1$ and equipped with an antiholomorphic involution σ covering the antipodal map on $\mathbb{CP}^1$.
- M with its hypercomplex structure can be recovered as a family of σ-invariant sections of $Z \to \mathbb{CP}^1$ with normal bundle splitting as $O(1) \otimes \mathbb{C}^n$.
- The basic example is $M = \mathbb{H}^n$, where the twistor space is simply the total space of $O(1) \otimes \mathbb{C}^n$.
- The twistor space Z of $S^1 \times \mathbb{R}^3$ is the total space of a nonalgebraic line bundle L over $T\mathbb{P}^1$ with the zero section removed.

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Higher degree curves

- It turns out that hyperkähler or hypercomplex manifolds can also be obtained from higher degree curves.

Theorem

Let $\pi : Z \to \mathbb{P}^1$ be the twistor space of 4-dimensional hyperkähler manifold. Then the subset $\mathcal{M}_d$ of the smooth locus of the Hilbert scheme of σ-invariant curves of degree d, satisfying $H^*(N_{C/Z}(-2)) = 0$, is, if nonempty, a hyperkähler manifold of real dimension $4d$.

- This is generalisation of a result of O. Nash, who showed that if we start with the twistor space of $S^1 \times \mathbb{R}^3$, then the hyperkähler manifold of stable curves of degree d in Z is the moduli space of $SU(2)$-monopoles of charge d with its natural hyperkähler metric.
- If we start with the twistor space of an ALE-space, we shall get complete hyperkähler manifolds (Seidel and Smith, Manolescu, Jackson). Similarly, if we start with the Taub-NUT metric, the Atiyah-Hitchin manifold or the resolution of the D_2-singularity.

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Projective curves

- What is almost completely unclear is the simplest case: the twistor space $Z = \mathbb{P}^3 - \mathbb{P}^1$ of $\mathbb{H}$, so that curves in Z are curves in the projective space not intersecting a fixed projective line.
- It is known that the parameter space $H_{d,g}$ of space curves with degree d and genus g contains smooth curves with $H^*(N_{C/\mathbb{P}^3}(-2)) = 0$ for any d greater than or equal to some $D(g)$ (e.g. $D(0) = 3$ and $D(1) = 5$). As soon as $H_{d,g}$ contains also a σ-*invariant* smooth curve with $H^*(N_{C/\mathbb{P}^3}(-2)) = 0$, we obtain a natural pseudo-hyperkähler structure on a submanifold of $H_{d,g}$.
  Can we compute these metrics? Are they complete?
  In the simplest case $g = 0, d = 3$, i.e. twisted cubics, the hyperkähler manifold has been identified as the flat $\mathbb{R}^{12}$ (thus complete).
- The case $g = 0, d = 4$ will be the starting point of the project.

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Objectives

We therefore want to investigate the global geometry of such moduli spaces of curves via twistor methods; more precisely we want to investigate the completeness and the asymptotic geometry of natural metrics on manifolds arising as smooth loci of Hilbert schemes of real algebraic curves (satisfying certain stability conditions) in complex (non-compact) manifolds, particularly in 3-folds.

The “geometry” here is not restricted to hypercomplex or hyperkähler. It includes quaternion-Kähler, and so called paraconformal and pluricomplex geometries.

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Research goals

- to obtain new examples of several interesting differential-geometric structures, including hyperkähler and quaternion-Kähler metrics and pluricomplex structures.
- to investigate global properties of these new examples, in particular their completeness.
- to study the asymptotic geometry and geometric compactifications of manifolds arising as such moduli spaces of curves via compactification of the relevant twistor space. We hope that this approach will have applications to computing L^2-cohomology for such manifolds, particularly those which which are physically relevant, such as monopole or Hitchin's moduli spaces.

Stability and instability of Einstein manifolds with prescribed asymptotic geometry

Klaus Kröncke

University of Hamburg

Kickoff meeting of the SPP 2026
University of Potsdam, November 10, 2017

Problem setting

Question

Let (M, g) be a Riemannian manifold which is Einstein (i.e. $\mathrm{Ric} = \lambda \cdot g$). Is it a stable fixed point of the Ricci flow

$$\dot{g} = -2\mathrm{Ric}_g$$

on the space of metrics modulo rescalings?

Known results

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  - The Euclidean Schwarzschild metric is unstable (Takahashi).

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We want to understand the

- stability problem unter Ricci flow in the case of Einstein manifolds with prescribed asymptotics (ALE, ALF, ALG, ALH, asymptotically (locally) conical, asymptotic products,...)

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- moduli space of Einstein manifolds in these cases (smoothness?).
- spectrum of the linearized operator Δ_L in some explicit cases (e.g. generalized (AdS-)Riemannian Schwarzschild metrics).

Subproblems and related problems

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- Long-time behaviour of the heat kernel of Δ_L.
- Positive scalar curvature rigidity of noncompact Ricci-flat manifolds.
- Positive mass problems for prescribed asymptotics.
- Defining entropies for the Ricci flow in the noncompact case.

References

- A. Deruelle, K. Kröncke, Stability of ALE Ricci-flat manifolds under Ricci flow, preprint arXiv:1707.09919

References

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Thank you for your attention!

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GEOMETRY AT INFINITY

Curvature Flows without Singularities

Wolfgang Maurer

(Oliver C. Schnürer)

University of Konstanz

09/10.11.2017

SPP 2026: Kick-Off Meeting

Mean Curvature Flow

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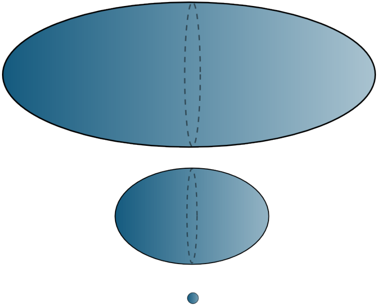
Let M^n be an n-dimensional manifold and let $X\colon M^n \times [0, T) \to \mathbb{R}^{n+1}$ be such that $X(\cdot, t)$ is an immersion for all t.

$$\frac{\mathrm{d}}{\mathrm{d}t}X = -H\,\nu \equiv -\text{mean curvature} \cdot \text{normal} \qquad \text{(Mean Curvature Flow (MCF))}$$



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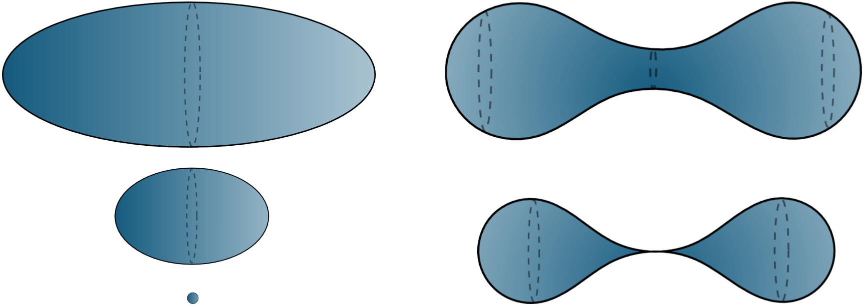

Graphical Mean Curvature Flow

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Let $\Omega \subset \mathbb{R}^n$ be open and let $u\colon \Omega \times [0,T) \to \mathbb{R}$.
Then graph $u(\cdot,t)$ moves by MCF if and only if $u(x,t)$ solves the quasi-linear parabolic PDE

$$\text{-normal velocity} \equiv \frac{\partial_t u}{\sqrt{1+|\nabla_x u|^2}} = \operatorname{div}\left(\frac{\nabla_x u}{\sqrt{1+|\nabla_x u|^2}}\right) \equiv \text{mean curvature.} \tag{Graphical MCF}$$

Starting from a closed hypersurface, singularities will always form.
But the graphical case is better behaved ...
(e. g. K. Ecker, G. Huisken, *Interior estimates for hypersurfaces moving by mean curvature*, JDG 1991)

MCF without Singularities

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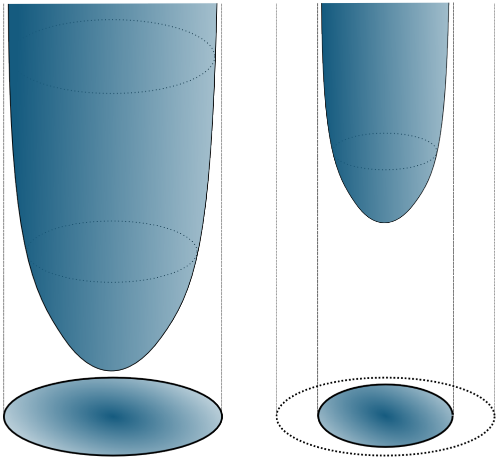

MCF without Singularities

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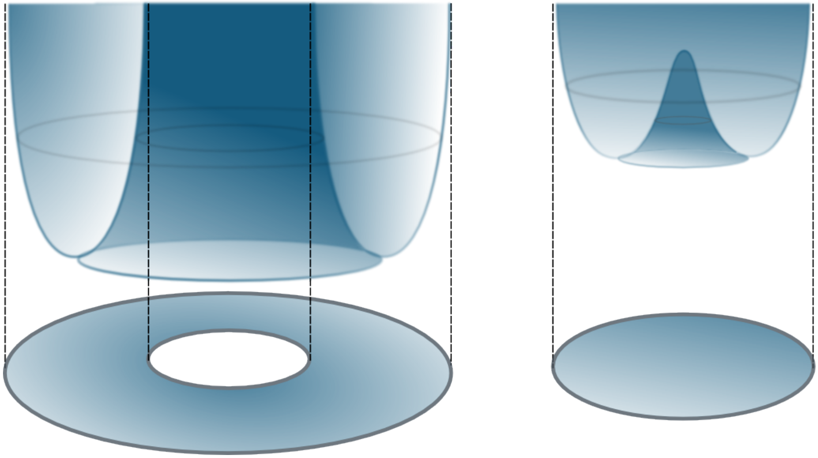

MCF without Singularities

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Theorem (Existence) (M. Sáez, O. Schnürer; W. Maurer)

For given initial data, there exists a space-time domain Ω and a function $u\colon \Omega \to \mathbb{R}$ solving

$$\frac{\partial_t u}{\sqrt{1+|\nabla_x u|^2}} = \operatorname{div}\left(\frac{\nabla_x u}{\sqrt{1+|\nabla_x u|^2}}\right) \quad \text{in } \Omega$$

with the given initial data and $|u| \to \infty$ at $\partial\Omega$ for $t > 0$.

The domain of definition Ω is part of the solution.
Regarding the time-slices Ω_t, we have the following result:

Theorem (Weak interpretation) (M. Sáez, O. Schnürer; W. Maurer)

The boundaries $\partial\Omega_t$ solve a weak version of mean curvature flow.

Objectives

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- Prove uniform curvature estimates dependent on the geometry of $\partial\Omega_t$
- Show that the solution is smoothly asymptotic to the cylinder $\partial\Omega_t \times \mathbb{R}$
- Understand the problem of uniqueness and non-uniqueness
- Behaviour at fattening
- Construction of translating solutions
- Understand singularities at infinity

Further objectives

- Study the corresponding Neumann boundary value problem
- Fully nonlinear curvature flows (more general normal velocities)
- Mean curvature flow without singularities on manifolds

Problems and Methods

Problems

- Non-compactness of the setting
- Existing estimates are height-dependent
  But we expect better behaviour at non-singular times

Methods

- Regularity theory for MCF
- Maximum principles / avoidance principles / a priori estimates
- Monotonicity formulas / entropy estimates
- Rescaling arguments / arguing by contradiction
- Theory of weak solutions
- Local graph representations

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Thank you for your attention!

Theorems in Detail

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Theorem (M. Sáez, O. Schnürer)

For $u_0 \colon A \to \mathbb{R}$, $A \subset \mathbb{R}^n$ open, bounded from below, locally Lipschitz and proper, there exists (Ω, u), where $\Omega \subset \mathbb{R}^n \times [0, \infty)$ is open and $u \colon \Omega \to \mathbb{R}$ proper, such that

$$\begin{cases} \dfrac{\partial_t u}{\sqrt{1 + |\nabla_x u|^2}} = \operatorname{div}\left(\dfrac{\nabla_x u}{\sqrt{1 + |\nabla_x u|^2}}\right) & \text{in } \Omega, \\ u(\cdot, 0) = u_0 & \text{in } A. \end{cases}$$

Theorem (M. Sáez, O. Schnürer)

In the above setting, if u_0 is bounded from below and if the level-set flow of $\partial\Omega_0$ does not fatten, then the measure theoretic boundary $\partial^\mu \Omega_t$ coincides with the level-set flow starting from $\partial\Omega_0$.

Theorems in Detail

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Theorem (W. Maurer)

Let $\Omega \subset \mathbb{R}^n$ be open, smooth ($\partial\Omega \in C^\infty$) and mean convex ($H[\partial\Omega] \geq 0$). Let $u_0 \colon \Omega \to [-\infty, \infty]$ be continuous and locally Lipschitz-continuous on $\{|u_0| < \infty\}$. Then there exists a continuous function $u \colon \Omega \times [0, \infty) \to [-\infty, \infty]$, smooth on $\{|u| < \infty,\ t > 0\}$, solving

$$\begin{cases} \dfrac{\partial_t u}{\sqrt{1 + |\nabla_x u|^2}} = \operatorname{div}\left(\dfrac{\nabla_x u}{\sqrt{1 + |\nabla_x u|^2}}\right) & \text{in } \{|u| < \infty,\ t > 0\}, \\ u(x,t) = u_0(x) & \text{for } t = 0 \text{ or } x \in \partial\Omega. \end{cases}$$

Moreover the set $\{|u| < \infty\}$ is a weak solution of mean curvature flow in the sense of barriers, i. e. any classical solution starting inside, stays inside, and any classical solution starting outside, stays outside.

Solutions to Ricci flow with scalar curvature bounded in L^p

Jiawei Liu and Miles Simon
(University of Magdeburg, Germany)

Kick Off Meeting , 9. and 10. Nov., 2017, Potsdam

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DFG Deutsche Forschungsgemeinschaft

The Ricci Flow equation

M^n smooth, closed (ie. $\partial M = \emptyset$ and M is compact) manifold, $dim(M^n) = n$.

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$(M^n, g(t))_{t \in [0,T)}$ solves Ricci-Flow (RF) (R. Hamilton, '82) if g is smooth in space and time and solves

$$\frac{\partial}{\partial t} g(x,t) = -2\mathrm{Rc}(g(\cdot,t))(x)$$

for all $x \in M$ for all $t \in [0, T)$.

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Theorem 1 (Hamilton '82)

If (M^n, g_0) *is closed then there exists a unique smooth,* *maximal* *solution* $(M, g(t))_{t\in[0,T)}$ *to Ricci flow for some* $T \in [0, \infty]$. *If* $T < \infty$, *then* $\sup_M |\mathrm{Rm}(g(t))| \to \infty$ *as* $t \nearrow T$.

In case $T < \infty$, singularities are forming as $t \nearrow T$.

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Examples:

$(M^n, g(0)) = (S^n, \gamma)$, where γ is the standard metric with $\mathrm{Sec} = 1$. Then $(S^n, g(t))_{t\in[0,T)} = (S^n, (1-2(n-1)t)\gamma), t \in [0, T = \frac{1}{2(n-1)})$

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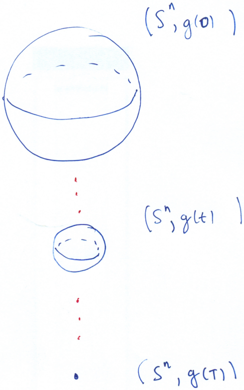

Neckpinching: Si. 2000, Angenent/Knopf 2002.

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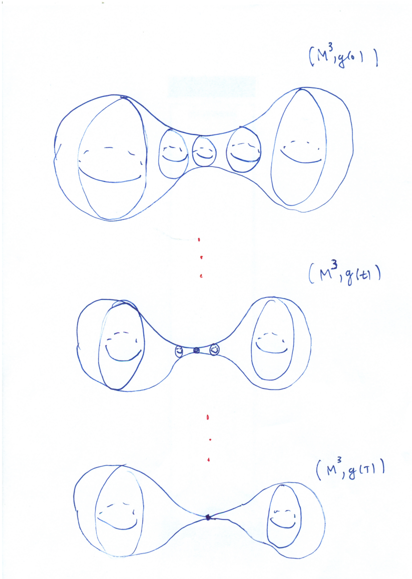

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In higher dimensions there is far less known about the possible types of singularities that can occur for arbitrary solutions.

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better understanding the types of finite time singularities that can occur in the case that $n \geq 4$

For the rest of the talk we restrict to the case that T is a possible finite singular time, that is $0 < T < \infty$. Estimates which give some information on the possible types of finite time singularities that can occur in dimenion $n = 4$ are known:

Theorem 2 (Si. ,'15: Basic Integral Estimate)

Let $(M^4, g(t))_{t\in[0,T)}$ *be an arbitrary smooth solution to Ricci flow on a closed oriented* 4*-manifold* M^4 *and assume that the scalar curvature satisfies* $\inf_M \mathrm{R}(\cdot, 0) > -1$ *at time zero. Then*

$$\int_M \frac{|\mathrm{Rc}|^2(\cdot, S)}{(\mathrm{R}(\cdot, S)+2)} d\mu_{g(S)} + \int_0^S \int_M \frac{|\mathrm{Rc}|^4(\cdot, t)}{(\mathrm{R}(\cdot, t)+2)^2} d\mu_{g(t)} dt$$

$$\leq e^{32S}\left(c_0(M, g_0) + 2^{10}\int_0^S \int_M \mathrm{R}^2(\cdot, t) d\mu_{g(t)}) dt\right)$$

for all $S < T$*, where* c_0 *depends only on the initial Riemannian manifold.*

We restrict the class of solutions to those with

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After scaling once at time zero, we can assume without loss of generality that $\sup_{M\times[0,T)} |\mathrm{R}| < 1$ due to the scaling properties of Ricci flow. Inserting this and the Gauß-Bonnet Theorem in $4D$,

$$\int_M |\mathrm{Rm}|^2 d\mu_g = 32\pi^2\chi + \int_M (4|\mathrm{Rc}|^2 - \mathrm{R}^2) d\mu_g,$$

into the Basic Integral Estimate, we get the much stronger estimates...

$$\int_M |\mathrm{Rm}|^2(\cdot, S)\,d\mu_{g(S)} \le \hat{c}_0(M, g_0)e^{32T} \tag{1}$$

and

$$\int_0^T \int_M |\mathrm{Rc}|^4(\cdot, t)\,d\mu_{g(t)}\,dt \le \hat{c}_0(M, g_0)e^{32T} \tag{2}$$

for all $S < T$.

$$\int_M |\mathrm{Rm}|^2(\cdot, S)d\mu_{g(S)} \leq \hat{c}_0(M, g_0)e^{32T} \qquad (1)$$

and

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for all $S < T$. (1) and other estimates in the case of $\sup_{M\times[0,T)} |\mathrm{R}| \leq 1$, were also obtained by R.Bamler/Q.Zhang ('15) independently using different methods (heat kernel methods).

In earlier papers of Q. Zhang '07,'13, R. Ye,'07, X. Chen/ B. Wang '13 it was shown for general n and closed M^n, using monotone quantities introduced by Perelman, that $\sup_{M\times[0,T)} |\mathrm{R}| < 1$ and $T < \infty$ also implies a uniform Sobolev inequality (uniform in the sense that the constants involved only depend on T and (M, g_0))

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$$\sigma_1 \leq \frac{\mathrm{vol}(B_{g(t)}(x, r))}{r^n} \leq \sigma_0$$

for all $x \in M, 0 \leq t < T$ and $0 < r \leq 1$, where σ_1, σ_2 are fixed positive constants.

Using these facts (on the Sobolev Inequality and the volumes of evolving balls),

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$$\int_M |\mathrm{Rm}|^2(\cdot, S)d\mu_{g(S)} \leq \hat{c}_0(M, g_0)e^{32T} \qquad (1),$$

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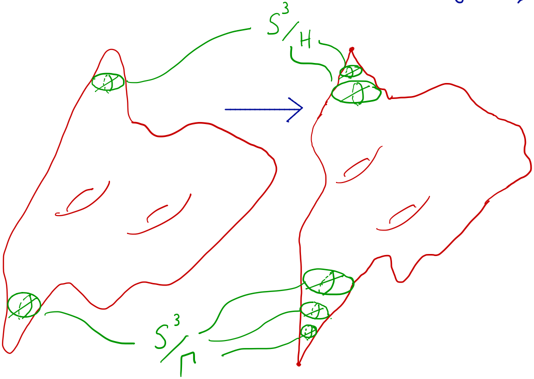
for all $S < T$, and carefully analysing the regions where the full Riemannian curvature concentrates, respectively doesn't concentrate, in the L^2 sense as $t \nearrow T$, we see...

Theorem 3 (Si. '15, R. Bamler+Q. Zhang '15)

Let $(M^4, g(t))_{t\in[0,T)}, T < \infty$, *be closed, oriented solution to (RF), and assume* $\sup_{M\times[0,T)} |\mathrm{R}| < \infty$. *Then* $(M, d(g(t)) \to (X, d_X)$ *as* $t \nearrow T$ *in the Gromov-Hasudorff sense, where* (X, d_X) *is a Riemannian* C^0 *orbifold with finitely many orbifold points* $\{p_1, \dots, p_K\}$. *There is a smooth diffeomorphism* $f : V \subset M \to X\backslash\{p_1, \dots, p_K\}$ *and here the convergence is smooth: There exists a smooth Riemannian metric* γ *on* $X\backslash\{p_1, \dots, p_K\}$ *s.t.* $f_*(g(t)) \to \gamma$ *in the* C^∞_{loc} *sense, and the distance induced on* X *by* γ *is* d_X.

$$(M^4, g(t)) \xrightarrow{t \to T} (M^4, g(T))$$



$H, \Gamma \subseteq O(4)$, finite subgroups

Theorem 4 (Si. '15)

It is possible to continue the flow from (X, d_X) *for a short time using the orbifold Ricci flow. Here, one obtains a smooth solution* $(X, h(t))_{t\in(T,T+\alpha)}$ *for some* $0 < \alpha < \infty$ *such that* $(X, d(h(t)) \to (X, d_X)$ *in the Gromov-Hausdorff sense as* $t \searrow T$.

The goals of this project are:

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(ii) In the case $n = 4$ use the Basic Integral Estimate and the estimates from (i) to investigate and understand better the structure of possible Gromov-Hausdorff limits as $t \nearrow T$ of our solutions.

Thank you for your attention!

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Nonlinear evolution equations on singular manifolds

Nikolaos Roidos
SCHR 319/9-1
Principal Investigator: Elmar Schrohe
Institut für Analysis, Leibniz Universität Hannover

Kick-Off Meeting
Potsdam 2017

Manifolds with conical singularities

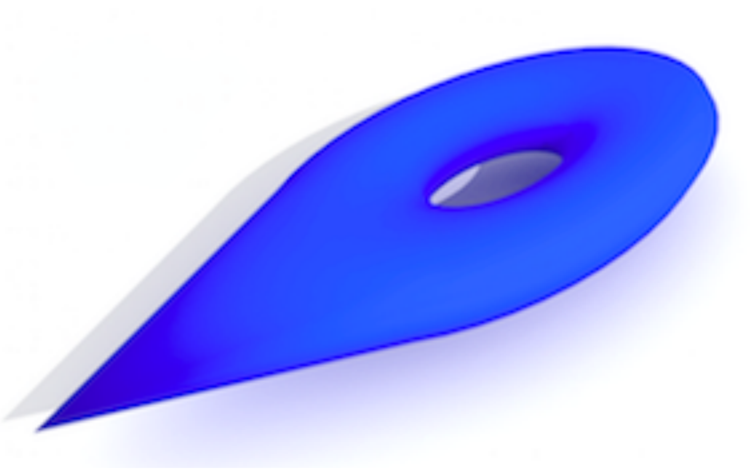
We model a manifold with conical singularities by a smooth compact n-dimensional, $n \geq 2$, manifold $\mathcal{B}$ with boundary $\partial\mathcal{B}$ endowed with a Riemannian metric g such that

$$g = dx^2 + x^2 h(x) \quad \text{on} \quad [0,1) \times \partial\mathcal{B},$$

where $x \in [0,1)$ and $h(x)$ is a smooth (up to $x = 0$) family of non-degenerate Riemannian metrics on $\partial\mathcal{B}$.

- $\partial\mathcal{B}$ can be disconnected.
- $\{0\} \times \partial\mathcal{B}$ corresponds to the conical tips.
- Denote $\mathbb{B} = (\mathcal{B}, g)$ and call it *conic manifold*.
- Consider also *manifolds with boundary and conical singularities*.

A conic manifold



The Laplacian on a conic manifold

The associated Laplacian on $(0,1) \times \partial\mathcal{B}$ has the conically degenerate form:

$$\Delta = \frac{1}{x^2}\Big((x\partial_x)^2 + (n-1+\frac{x\partial_x \det[h(x)]}{2\det[h(x)]})(x\partial_x) + \Delta_{h(x)}\Big),$$

where $\Delta_{h(x)}$ is the Laplacian on $\{x\} \times \partial\mathcal{B}$ induced by the metric $h(x)$.

- Impose Dirichlet or Neumann boundary conditions in the case of a manifold with boundary and conical singularities.
- Regard Δ as a *cone differential operator*.
- Make use of the *cone calculus*.

The porous medium equation

The *porous medium equation* (PME) is the parabolic diffusion equation

$$\begin{aligned} u'(t) - \Delta(u^m(t)) &= 0, \quad t > 0, \\ u(0) &= u_0, \end{aligned}$$

where $m > 0$. It describes the flow of a *gas* in a porous medium; u is the density distribution and Δ is the (negative) Laplacian.

The Cahn-Hilliard equation

The *Cahn-Hilliard equation* is the following higher order semilinear problem

$$\begin{aligned} u'(t) + \Delta^2 u(t) + \Delta\big(u(t) - u^3(t)\big) &= 0, \quad t > 0, \\ u(0) &= u_0. \end{aligned}$$

This is a phase-field or diffuse interface equation which models phase separation of a binary mixture. Here u denotes the concentration difference of the components; the sets where $u = \pm 1$ correspond to domains of pure phases.

Objectives

- Show existence, uniqueness and maximal regularity for the short and long time solution.
- Obtain precise information concerning the asymptotic behavior of the solution close to the singularity and show a relation between the evolution and the local geometry.
- Develop the cone calculus in the direction of the regularity theory for PDEs.

Difficulties

- Degeneracy and several closed extensions for Δ.
- Lack of basic PDEs-machinery (e.g. elliptic regularity, comparison principle, gradient estimates etc) and failure of standard techniques for existence and regularity theory.
- The cone calculus, at least for the case of boundary value problems, is not well developed.

Elements of the cone calculus: Mellin-Sobolev spaces

- The Laplacian acts naturally on the scales of *Mellin-Sobolev spaces* $\mathcal{H}_p^{s,\gamma}(\mathbb{B})$, $s, \gamma \in \mathbb{R}$, $p \in (1, \infty)$.
- e.g. if $s \in \mathbb{N}$, the space $\mathcal{H}_p^{s,\gamma}(\mathbb{B})$, $\gamma \in \mathbb{R}$, $p \in (1, \infty)$, is the space of all functions u in $H^s_{loc}(\mathrm{int}\,\mathbb{B})$ such that near the boundary

$$x^{\frac{n}{2}-\gamma}(x\partial_x)^k \partial_y^\alpha(\omega(x)u(x,y)) \in L^p\left([0,1) \times \partial\mathcal{B}, \sqrt{\det[h(x)]}\frac{dx}{x}dy\right), \quad k + |\alpha| \le s,$$

where ω is a cut-off function on the collar part $[0, 1) \times \partial\mathcal{B}$.

... closed extensions of the Laplacian

- $\Delta : C_c^\infty(\operatorname{int}\mathbb{B}) \mapsto \mathcal{H}_p^{s,\gamma}(\mathbb{B})$, $s, \gamma \in \mathbb{R}$, $p \in (1, \infty)$, as a second order $\mathbb{B}$-*elliptic cone differential operator* admits several closed extensions, namely:

$$\mathcal{D}(\underline{\Delta}_{\max}) = \mathcal{H}_p^{s+2,\gamma+2}(\mathbb{B}) \oplus \mathcal{E}.$$

- $\mathcal{E}$ is a finite dimensional space consisting of smooth functions on $\mathbb{B}$ that in local coordinates close to $\{0\} \times \partial\mathcal{B}$ are of the form

$$c_0(y)x^{\rho_0} + c_1(y)x^{\rho_1}\log(x), \quad (x, y) \in (0, 1) \times \partial\mathcal{B},$$

  where $c_0, c_1 \in C^\infty(\partial\mathbb{B})$. The powers $\rho_0, \rho_1 \in \mathbb{C}$ are determined explicitly by the metric $h(x)$.
- We choose

$$\mathcal{D}(\underline{\Delta}) = \mathcal{H}_p^{s+2,\gamma+2}(\mathbb{B}) \oplus \mathbb{C}.$$

Recent results in the boundaryless case

(D) (choice of the data)

Denote by λ_1 the greatest non-zero eigenvalue of $\Delta_{h(0)}$ and choose

$$\frac{n}{2} + \frac{2}{q} - 2 < \gamma < \min\left\{ -1 + \sqrt{\left(\frac{n}{2} - 1\right)^2 - \lambda_1}, \frac{n}{2} \right\}$$

with

$$\frac{n}{p} + \frac{2}{q} < 1, \quad p, q \in (1, \infty).$$

Furthermore, let

$$s > \max\left\{ -1 + \frac{n}{p} + \frac{2}{q}, -\frac{2}{q} \right\}.$$

Long time existence for solution of the PME

Theorem ($*$)

Let s, γ, p, q be chosen as in (D) and $u_0 \in (\mathcal{H}_p^{s+2,\gamma+2}(\mathbb{B}) \oplus \mathbb{C}, \mathcal{H}_p^{s,\gamma}(\mathbb{B}))_{\frac{1}{q},q}$ *such that* $c_1 \le u_0 \le c_2$, $c_1, c_2 > 0$. *Then, for each* $T > 0$ *there exists a unique*

$$\begin{aligned} u \in\ & C^1((0,T];\mathcal{H}_p^{s,\gamma}(\mathbb{B})) \cap C((0,T];\mathcal{H}_p^{s+2,\gamma+2}(\mathbb{B}) \oplus \mathbb{C}) \\ & \cap C^{\delta}([0,T];\mathcal{H}_p^{s+2-\frac{2}{q}-2\delta-\varepsilon,\gamma+2-\frac{2}{q}-2\delta-\varepsilon}(\mathbb{B}) \oplus \mathbb{C}) \\ & \cap C^{1+\delta}([0,T];\mathcal{H}_p^{s-\frac{2}{q}-2\delta-\varepsilon,\gamma-\frac{2}{q}-2\delta-\varepsilon}(\mathbb{B})) \\ & \cap W^{1,q}(0,T;\mathcal{H}_p^{s,\gamma}(\mathbb{B})) \cap L^q(0,T;\mathcal{H}_p^{s+2,\gamma+2}(\mathbb{B}) \oplus \mathbb{C}) \\ & \cap C([0,T];\mathcal{H}_p^{s+2-\frac{2}{q}-\varepsilon,\gamma+2-\frac{2}{q}-\varepsilon}(\mathbb{B}) \oplus \mathbb{C}) \end{aligned}$$

for all $\varepsilon > 0$ *and* $\delta \in (0, \frac{1}{2}\min\{2 - \frac{n}{p} - \frac{2}{q}, \gamma + 2 - \frac{n}{2} - \frac{2}{q}\})$, *solving the PME on* $[0,T] \times \mathbb{B}$, *which also satisfies* $c_1 \le u \le c_2$.

($*$) N. Roidos, E. Schrohe. *Smoothness and long time existence for solutions of the porous medium equation on manifolds with conical singularities*. [arXiv:1708.07542].

Asymptotic behavior close to the singularity

- By the usual embedding of the maximal L^q-regularity space we have that

$$u \in C([0,T]; \mathcal{H}_p^{s+2-\frac{2}{q}-\varepsilon,\gamma+2-\frac{2}{q}-\varepsilon}(\mathbb{B}) \oplus \mathbb{C}) \hookrightarrow C([0,T]; C(\mathbb{B}))$$

  for all $\varepsilon > 0$.

- By Mellin-Sobolev embedding we obtain

$$|u_{\mathcal{H}}(t)| \leq Cx^{\gamma+2-\frac{2}{q}-\frac{n}{2}-\varepsilon} \|u_{\mathcal{H}}(t)\|_{\mathcal{H}_p^{s+2-\frac{2}{q}-\varepsilon,\gamma+2-\frac{2}{q}-\varepsilon}(\mathbb{B})}, \quad t \in [0,T],$$

  for all $\varepsilon > 0$ and some constant $C > 0$ depending on n and p, where $u_{\mathcal{H}}$ is the Mellin-Sobolev part of u, i.e. $u(t) = u_{\mathcal{H}}(t) \oplus c(t)$, with $c(t) \in \mathbb{C}$, $t \in [0,T]$.

Smoothness for solutions of the PME

Theorem (∗)

For the unique solution u of the PME on $[0,T]\times\mathbb{B}$, $T>0$, we additionally have that

$$u \in \bigcap_{\nu,\varepsilon>0} C^1((0,T];\mathcal{H}_p^{\nu,\gamma}(\mathbb{B}))\cap C((0,T];\mathcal{H}_p^{\nu,\gamma+2}(\mathbb{B})\oplus\mathbb{C})$$
$$\cap\, C^{\delta}((0,T];\mathcal{H}_p^{\nu,\gamma+2-2\delta-\frac{2}{q}-\varepsilon}(\mathbb{B})\oplus\mathbb{C})$$
$$\cap\, C^{1+\delta}((0,T];\mathcal{H}_p^{\nu,\gamma-2\delta-\frac{2}{q}-\varepsilon}(\mathbb{B}))$$

for each $\delta\in(0,\frac{1}{2}\min\{2-\frac{n}{p}-\frac{2}{q},\gamma+2-\frac{n}{2}-\frac{2}{q}\})$*. Furthermore,*

$$u\in\bigcap_{\nu>0} C^k((0,T];\mathcal{H}_p^{\nu,\gamma-2(k-1)}(\mathbb{B}))\quad \textit{for all}\quad k\in\mathbb{N}.$$

(∗) N. Roidos, E. Schrohe. *Smoothness and long time existence for solutions of the porous medium equation on manifolds with conical singularities*. [arXiv:1708.07542].

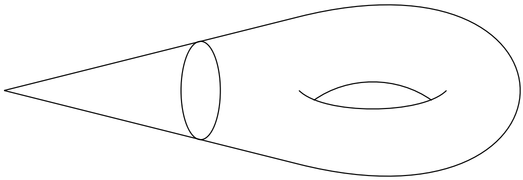
Thank you for your attention!

Geometric analysis on singular spaces

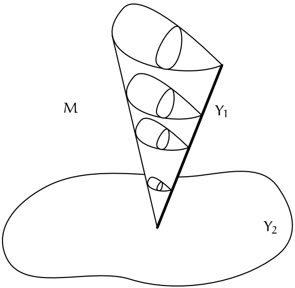
Boris Vertman and Matthias Lesch

Spectral geometry, index theory and geometric flows on singular spaces

10. November 2017



1. Stratified spaces and fundamental questions



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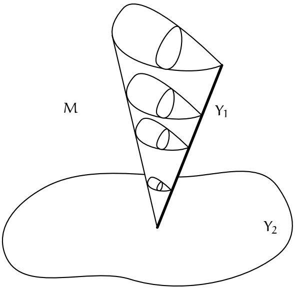

- Spectral geometry on stratified spaces
- Index theory, eta and Cheeger - Gromov rho invariants
- Geometric flows, Ricci, Yamabe and mean curvature flows

2. Research objectives in spectral geometry

$$\zeta(s,\Delta) := \frac{1}{\Gamma(s)} \int_0^\infty t^{s-1} \left(\mathrm{Tr}\left(e^{-t\Delta}\right) - \dim\ker\Delta \right) dt.$$

2. Research objectives in spectral geometry

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- heat-trace asymptotics for Laplacians on stratified spaces,
- Bergman kernel asymptotics and quantum Hall effect,
- Spectral geometry on edges with variable indicial roots,
- Cheeger-Müller Theorem on spaces with

  even codimension singularities.

Some preliminary results in spectral geometry

Some preliminary results in spectral geometry

Theorem (jointly with Rafe Mazzeo)

Analytic torsion of an admissible edge space (M, g^M) with edge Y

$$T(M, g^M) := \exp\left(\frac{1}{2}\sum_{k=0}^{\dim M}(-1)^k \cdot k \cdot \zeta'(0, \Delta_k)\right).$$

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Analytic torsion of an admissible edge space (M, g^M) with edge Y

$$T(M, g^M) := \exp\left(\frac{1}{2}\sum_{k=0}^{\dim M}(-1)^k \cdot k \cdot \zeta'(0, \Delta_k)\right).$$

1. exists and depends only on the leading order of g^M near ∂M,
2. if $\dim Y$ is even, g^M is even in the defining function of Y, then $T(M, g^M)$ is independent of the choice of g^M

3. Research objectives in index theory

$$\text{index } \mathbb{D} = \left(\int_M a_0 + \int_Y b_0 \right) - \frac{\dim \ker D + \eta(D)}{2}.$$

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- Atiyah-Patodi-Singer theorem on stratified spaces,
- Cheeger-Gromov invariant and signature of strat. spaces,
- space of bordism classes of metrics with $\mathrm{scal}(g) > 0$.

Some preliminary results in index theory

Some preliminary results in index theory

Theorem (jointly with Paolo Piazza)

Assume the edge space M is boundary of an edge space N and the (signature or spin) Dirac operator $\mathbb{D}$ is of the form

$$\mathbb{D} = \sigma \left(\frac{\partial}{\partial u} + D \right),$$

near $M = \partial N$, satisfying spectral Witt condition.

Some preliminary results in index theory

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$$\mathbb{D} = \sigma \left(\frac{\partial}{\partial u} + D \right),$$

near $M = \partial N$, satisfying spectral Witt condition. Then[a]

1. $\mathbb{D}_{\mathrm{APS}}$ is Fredholm, $\eta(D)$ is well-defined
2. index $\mathbb{D} = (\int_M a_0 + \int_Y b_0) - (\dim\ker D + \eta(D))/2$,
3. The statement carries over to Galois coverings.

[a] under some dimensional restrictions

4. Research objectives in geometric flows

$$\frac{\partial}{\partial t} g^M(t) = -2 \, \mathrm{Ric}\left(g^M(t)\right), \quad g^M(0) = g_0^M.$$

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- Mean curvature flow on spaces with conical singularities,
- Stability of Ricci flow near Ricci flat conical metrics,
- Ricci flow of a metric with positive curvature tensor.

Some preliminary results in geometric flows

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Theorem (jointly with Eric Bahuaud)

Suppose (M, g_0) is an feasible edge space with $\dim M \geq 3$ such that $\mathrm{scal}(g_0) < 0$ is of some Hölder regularity. Then the

volume-normalized Yamabe flow

$$\partial_t g(t) = \Big(\rho(t) - \mathrm{scal}(g(t)) \Big) \cdot g(t), \;\; g(0) = g_0,$$

$$\rho(g) = \mathrm{vol}(g(t))^{-1} \int_M \mathrm{scal}(g(t)) \, \mathrm{dvol}_{g(t)} \, .$$

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$$\rho(g) = \mathrm{vol}(g(t))^{-1} \int_M \mathrm{scal}(g(t)) \, \mathrm{dvol}_{g(t)} \, .$$

1. exists for all times within the space of regular edge metrics,
2. $g(t) \to g^*$ as $t \to \infty$ with and $\mathrm{scal}(g(^*)) \equiv \mathrm{const} < 0$.

Thank you for your attention!

Secondary invariants for foliations

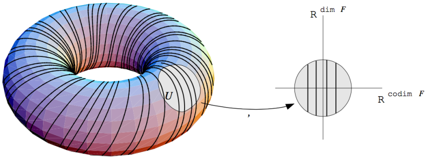
Sara Azzali (Potsdam) and Sebastian Goette (Freiburg)

Kick-Off-Meeting "Geometry at infinity"
Potsdam, November 9-10, 2017

SPP 2026
GEOMETRY AT INFINITY

Geometric setting

- A foliation is a decomposition of a closed manifold M into immersed submanifolds called the leaves. The leaves are noncompact but have bounded geometry.
  Example: a torus $M = \mathbb{T}^k \times \mathbb{T}^k \mathbb{R}^k \times \mathbb{R}^k / \mathbb{Z}^k \times \mathbb{Z}^k$ *with foliation* $\mathcal{F}$ *given by the product of affine lines with slopes* $\alpha_1, \cdots, \alpha_k \in \mathbb{R} \setminus \mathbb{Q}$



picture: $k = 1$

  If $\alpha_i \in \mathbb{Q}$, *then* $p : M \to M/\mathcal{F} = \mathbb{T}^k$ *is a proper submersion*

- Leafwise operators: We will mainly deal with (Dirac type) operators acting leafwise on M.
  For example: the family $D = (d_L + d_L^*)_{L \in M/\mathcal{F}}$ *the Hodge–de Rham operator along the leaves* L

Primary versus secondary invariants

Let $p : M \to B$ be a proper submersion, D the fibrewise signature operator

- Theorem [Bismut]: The heat operator $e^{-\mathbb{A}_t^2}$ interpolates between at small time the topological index and at large time the index bundle.

$$\underbrace{\int_{M/B} L(TZ)}_{\lim_{t\to 0} \mathrm{Str}(e^{-\mathbb{A}_t^2})} - \underbrace{\mathrm{ch}\big(\mathrm{Ker}\, D \ominus \mathrm{Ker}\, D^*\big)}_{\lim_{t\to \infty} \mathrm{Str}(e^{-\mathbb{A}_t^2})} = d \underbrace{\hat{\eta}\big(T^H M, g^{TZ}\big)}_{\text{Bismut–Cheeger eta form}} \in \Omega^*(B)$$

- In the context of index theory, a secondary invariant is a geometric term which interpolates between two realisation of an index class.

PRIMARY	SECONDARY
index class	eta form
flat bundles classes	torsion forms

Secondary invariants. Motivations

- Eta and torsion forms are higher analogues of the Atiyah–Patodi–Singer eta invariant and of the Ray–Singer torsion
- Adiabatic limits: if $p : M \to B$ is a proper submersion, deforming the metric in the normal direction, i.e. $g_\varepsilon = \frac{1}{\varepsilon^2} g^\perp \oplus g^{T\mathcal{F}}$, then [Bismut–Cheeger, Dai]

$$\lim_{\varepsilon \to 0} \eta(D^M_{g_\varepsilon}) = \int_B L(TB) \wedge \hat{\eta} + \eta(D_B) + \sum_{r \geq 2} \sigma_r$$

Geometric applications:

- Relative eta invariants (or more generally rho classes) allow to distinguish geometric structures [Botvinnik–Gilkey, Chang–Weinberger, ...]
- Torsion invariants can detect exotic smooth structures on fibre bundles [Igusa–Goette]

On foliations

- The space of leaves is replaced by the holonomy groupoid $\mathcal{G}$
  Example: $\mathcal{G} = (\mathbb{R}^k \times \mathbb{R}^k \times \mathbb{T}^k)/\mathbb{Z}^k$ *in the Kronecker foliation*
- The index class belongs to the K-theory of $C^*(\mathcal{G})$ [Connes–Skandalis]
- Heat operator approach generalises to foliations via
  - Haefliger forms [Heitsch, Lazarov, Benameur] (*Ex:* $\Omega^*(\mathbb{T}^k)$)
  - Noncommutative forms [Lott, Gorokhovsky] (*Ex:* $\Omega^*(\mathbb{T}^k, \mathcal{B}^\infty_{\mathbb{Z}^k})$)
- Theorem [Benameur–Heitsch]: in the Haefliger setting, if the Novikov–Shubin invariants are larger than $\frac{1}{2}\operatorname{codim}(\mathcal{F})$, then $\lim_{t\to\infty} \operatorname{Str} e^{-\mathbb{A}_t^2} = \operatorname{ch}(P_{\operatorname{Ker} D})$
- eta and torsion forms can be defined under similar (strong) assumptions

Large time limit, index class versus kernel bundle

Theorem (A., Goette, Schick 2013): in the L^2-setting of a family of normal coverings and for $D = d_{\tilde{Z}} + d^*_{\tilde{Z}}$ it is enough to require positive Novikov–Shubin invariants

- $\lim_{\to\infty} \mathrm{Str}_\Gamma(e^{-\mathbb{A}_t^2}) = \mathrm{ch}_\Gamma(P_{\mathrm{Ker}\, D})$
- L^2-eta and -torsion forms are well defined.

For more general operators:
Example [Benameur, Heitsch, Wahl]: *on the Kronecker foliation $(\mathbb{T}^k \times \mathbb{T}^k, \mathcal{F})$, $d_L + d_L^*$ with a nonflat twist,*

$$\mathrm{ch}(\mathrm{Ind}\, D) \neq 0 = \mathrm{ch}(P_{\mathrm{Ker}\, D})$$

Yet, the heat operator still represents $\mathrm{ch}(\mathrm{Ind}\, D)$.
Question: Can one compute the large time limit of $\mathrm{Str}\, e^{-\mathbb{A}_t^2}$ in a generalised sense so that it captures the whole global informations?

Questions

On the large time limit of $e^{-\mathbb{A}_t^2}$:

1. Can one interpret and compute the large time limit of $\operatorname{Str} e^{-\mathbb{A}_t^2}$ in a suitable "generalised" sense so that it captures the whole index informations?
2. In the particular case $D = d + d^*$, can Haefliger eta and torsion forms be defined without extra regularity assumption on the spectrum?

Noncommutative aspects:

3. Can one provide refinements using noncommutative forms?

Adiabatic limits:

4. On a Riemannian foliation, consider the deformation $g_\varepsilon = \frac{1}{\varepsilon^2} g^\perp \oplus g^{T\mathcal{F}}$. What is $\lim_{\varepsilon \to 0} \eta(D^M_{g_\varepsilon})$?

Torsion forms and differentiable structures:

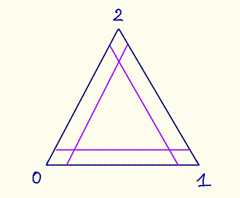
5. Once a Haefliger torsion form is defined, does it make sense to generalise the *arc de triomph* construction of Igusa–Goette to foliations? Can it be used to detect "exotic" smooth foliations on a given M?

Methods

1. Can one compute the large time limit of $\mathrm{Str}(e^{-\mathbb{A}_t^2})$ in a suitable "generalised" sense so that it captures the whole global informations?
   On the Kronecker foliation above, the heat operator can easily be written. We are currently investigating the relation between its large time limit and generalised eigenvalues.
2. Define Haefliger eta and torsion forms with minimal assumptions
   If we can find a "von Neumann type" estimate for the Haefliger trace, then methods from [AGS] apply: terms in the Volterra expansion

$$\int_{\Delta^n} e^{-s_0 tD^2} R_t e^{-s_1 tD^2} \ldots\ldots R_t e^{-s_n tD^2} d^n(s_0, \ldots, s_n)$$

   can be estimated via a splitting of the simplices in small/large variables pieces.



Methods

3. Can one provide refinements using noncommutative forms? Using the noncommutative approach à la Gorokhovsky-Lott, there are some partial results [So, Su 2017] for noncommutative torsion forms using generalisations of our work [AGS].
4. What is $\lim_{\varepsilon\to 0}\eta(D^M_{g_\varepsilon})$?
   On Riemannian foliations. Start from the the works by Alvarez-Lopez and Kordyukov on adiabatic limits.

Thank you for your attention!

Spectral Analysis of Sub-Riemannian Structures

Wolfram Bauer,
Abdellah Laaroussi

Leibniz Universität, Hannover

Kick-Off meeting, SPP 2026 - Geometry at infinity

Potsdam, 09.-10. November 2017

Outline

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1. Sub-Riemannian structures and hypoelliptic operators

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3. Spectral analysis of geometrically defined operators
4. Heat kernel of (sub)-Laplacians on forms and SR limit.

Introduction: Sub-Riemannian Geometry

A sub-Riemannian geometry abstractly models motions under non-holonomic constraints.

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Definition: sub-Riemannian manifold

A SR-manifold is a triple $(M, \mathcal{H}, \langle\cdot,\cdot\rangle)$, where

- M is a smooth manifold (without boundary), $\dim M \geq 3$ and $\mathcal{H} \subset TM$ is a vector distribution.

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A SR-manifold is a triple $(M, \mathcal{H}, \langle \cdot, \cdot \rangle)$, where

- M is a smooth manifold (without boundary), $\dim M \geq 3$ and $\mathcal{H} \subset TM$ is a vector distribution.
- the bundel $\mathcal{H}$ is bracket generating of rank $k < \dim M$, i.e.

$$\mathrm{Lie}_x \mathcal{H} = T_x M$$

  with inner product $\langle \cdot, \cdot \rangle_x$ on $\mathcal{H}_x$ where $x \in M$.

Horizontal connectivity

On $(M, \mathcal{H}, \langle \cdot, \cdot \rangle)$ we consider *"horizontal objects"*:

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Example

A curve $\gamma : [0, T] \to M$ is called horizontal if $\dot{\gamma}(t) \in \mathcal{H}_{\gamma(t)}$ a.e.

$$\ell(\gamma) := \int_0^T \sqrt{\langle \dot{\gamma}(t), \dot{\gamma}(t) \rangle_{\gamma(t)}} dt = \textit{length of } \gamma.$$

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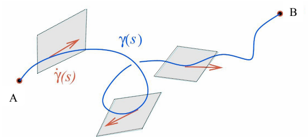
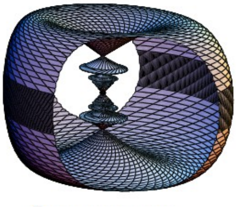
Theorem, (Chow-Rashevskii, 1938/39)

Let M be a connected SR-manifold. $A, B \in M$ can be joint by a horizontal path. M becomes a metric space with distance:

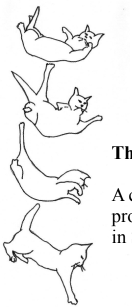
$$d_{CC}(A, B) := \inf \Big\{ \ell(\gamma) \ : \ \gamma(0) = A, \quad \gamma(1) = B, \quad \gamma(t) \text{ horizontal} \Big\}.$$



Distribution and horizontal curve



Front of SR geodesics at time T
(picture by: U. Boscain, D. Barilari)



The falling cat:

A connectivity
problem
in SR geometry

From Riemannian to SR-geometries

In some cases we may interpret a SR-geometry as the limit object of a family of Riemannian geometries, if the family of metrics blows up transversally to the defining distribution.

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Example:

Let $\mathbb{H}_3 \cong \mathbb{R}^3$ be the Heisenberg group with non-isotropic dilation

$$\delta_t(x, y, z) := \left(tx, ty, t^2 z\right) \qquad t > 0.$$

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$$\delta_t(x, y, z) := \big(tx, ty, t^2 z\big) \qquad t > 0.$$

Consider a family of Riemannian metrics $g_t := t^{-1}\delta_t^*(g)$, where g is left-invariant on $\mathbb{H}^3$. Then

$$d_{g_t} \longrightarrow d_{CC} \quad \textit{and} \quad t^{-1}\mathbb{H}^3 \longrightarrow \big(\mathbb{H}^3, d_{CC}\big) \quad (t \to 0).$$

(Convergence in the Gromov-Hausdorff sense).

Sub-Laplacian

Let $(M, \mathcal{H}, \langle \cdot, \cdot \rangle)$ be a SR-manifold and

$$[X_i \ : \ i = 1, \cdots, m] = \textit{local frame for } \mathcal{H}.$$

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Consider a smooth measure ω, a vector field X and a smooth function φ on M. Define

$$\mathcal{L}_X(\omega) = \mathrm{div}_\omega(X)\omega \qquad (\omega\text{-divergence})$$

$$\Big\langle \underbrace{\mathrm{grad}_{\mathcal{H}}(\varphi)}_{\in \mathcal{H}_x}, v \Big\rangle_x = d\varphi(v), \quad v \in \mathcal{H}_x \qquad \text{(horizontal gradient)}.$$

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Definition: Sub-Laplacian

The *intrinsic* Sub-Laplacian on $(M, \mathcal{H}, \langle \cdot, \cdot \rangle)$ is defined by

$$\Delta_{\mathrm{sub}} := \mathrm{div}_\omega \circ \mathrm{grad}_{\mathcal{H}} = \sum X_i^2 + (\text{first order term}).$$

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Consider a smooth measure ω, a vector field X and a smooth function φ on M. Define

$$\mathcal{L}_X(\omega) = \mathrm{div}_\omega(X)\omega \qquad (\omega\text{-divergence})$$

$$\Big\langle \underbrace{\mathrm{grad}_{\mathcal{H}}(\varphi)}_{\in \mathcal{H}_x}, v \Big\rangle_x = d\varphi(v), \quad v \in \mathcal{H}_x \qquad (\textit{horizontal gradient}).$$

Definition: Sub-Laplacian

The *intrinsic* Sub-Laplacian on $(M, \mathcal{H}, \langle\cdot,\cdot\rangle)$ is defined by

$$\Delta_{\mathrm{sub}} := \mathrm{div}_\omega \circ \mathrm{grad}_{\mathcal{H}} = \sum X_i^2 + (\text{first order term}).$$

Choose ω e.g. as the Popp volume on an equi-regular SR manifold.

Theorem (L. Hörmander, 1967)

Let $\Omega \subset \mathbb{R}^n$ be open. Consider C^∞-vector fields $[X_0, X_1, \cdots, X_m]$ on Ω with

Hypoellipticiy of "sum-of-squares operators"

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Then the second order differential operator

$$\mathcal{L} := \sum_{j=1}^{m} X_j^2 + X_0 + c \qquad c \in C^\infty(\Omega)$$

is hypoelliptic.

Problems (Selection)

1. Task: Existence and Construction of SR-geometries

Problem

Which "classical manifolds" (e.g. spheres, Lie groups, symmetric spaces) M carry a SR-structure? What is the Popp volume? What is the Sub-Laplacian?

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Example: Various cases are known in the literature, e.g.:

Theorem (B., Furutani, Iwasaki)

Trivializable SR-structures (i.e. $\mathcal{H}$ is trivial) on Euclidean spheres $\mathbb{S}^n$ only exist in dimensions $n = 3, 7, 15$. On $\mathbb{S}^7$ there are such structures of rank 4,5, and 6.

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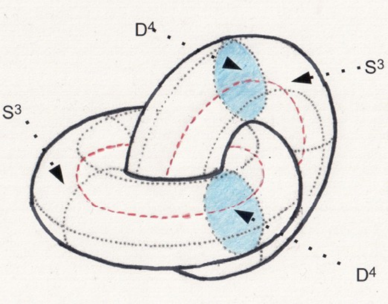
Question:

How about the existence of SR-structures on exotic 7-spheres?

Exotic 7-sphere



Exotic 7-Sphere of type (2,-1):

Clue 2 copies of $S^3 \times D^4$

along their boundaries via

$$S^3 \times \partial D^4 \longrightarrow S^3 \times \partial D^4$$
$$(u,v) \longmapsto (u, u^2 v u^{-1})$$

Gromoll-Meyer exotic 7-Sphere Σ^7_{GM}

Realization as the base of a Δ-principal bundle:

$$
\begin{tikzcd}
\Delta \arrow[dr, dotted] & \subset & G \arrow[dl, dotted] \\
& \mathrm{Sp}(2) \arrow[dl, "\pi_G"] \arrow[dr, "\pi_\Delta"'] & \\
\mathbb{S}^4 = \mathrm{Sp}(2)/G & & \mathrm{Sp}(2)/\Delta = \Sigma^7_{\mathrm{GM}}
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Theorem (W. B., K. Furutani, C. Iwasaki, 2016)

The bi-quotiend of compact groups induces a codim-3-SR-structure on the GM-realization Σ^7_{GM} of a type $(2, -1)$ exotic 7-sphere.

Problems to be considered

What is next?

- What is the Popp measure on Σ^7_{GM}?

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$$\begin{aligned} \mathcal{N}_{r,s} &:= V \oplus_{\perp} \mathbb{R}^{r,s} = \text{"}\textit{pseudo-}H\textit{-type Lie algebra}\text{"}, \\ G_{r,s} &:= \textit{connected, s.c. Lie group with Lie algebra } \mathcal{N}_{r,s}, \\ L &= \textit{lattice in } G_{r,s}. \end{aligned}$$

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$$\Delta_{\text{sub}}^{L \backslash G_{r,s}}.$$

Problem:

Systematically construct and classify all isospectral (w.r.t. the Sub-Laplacian) but non-diffeomorphic pseudo-H-type nilmanifolds.

Tools available

From the analytic point of view, there is an explicit trace formula:

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Theorem (W. B., K. Furutani, C. Iwasaki, 2015)

The heat trace of the sub-Laplacian is given by

$$\mathrm{Trace}\left(e^{t\Delta_{\mathrm{sub}}^{L\backslash G_{r,s}}}\right) = \frac{1}{(2\pi t)^{\frac{\dim G - \dim A}{2}}} \times \sum_{\mathbf{n}\in(L\cap A)^*} \sum_{\mu\in\mathbb{M}(\mathbf{n})} e^{-\frac{\langle\mu,\mu\rangle}{2t}} \sqrt{\det \frac{\Omega(2\pi\sqrt{-1}t\mathbf{n})}{\sinh\Omega(2\pi\sqrt{-1}t\mathbf{n})}}.$$

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From the algebraic - geometric point of view:

K. Furutani, I. Markina *Complete classification of pseudo H-type Lie algebras: I.* Geom. Dedicata 190 (2017), 23?51

(Sub)-Laplacian on differential forms over nilpotent groups:

"What can you hear at infinity of nilpotent groups?"
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Analytic objects in the sub-Riemannian limit

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Theorem (W.B., A. Froehly, et. al.)

There is an explicit integral form for the heat kernel of the Laplacian acting on 1-forms over the Heisenberg groups.

Thank you for your attention!

Analysis on spaces with fibred cusps

Daniel Grieser

(Carl von Ossietzky Universität Oldenburg)

10.11.2017

Kick-off meeting SPP *Geometry at infinity*

Overall goal

Goal of project

To develop analytic tools for analysing the natural geometric differential operators associated to fibred cusp metrics and their variants and to apply these tools to questions of global analysis and spectral theory.

Fibred cusps

Definition

A **fibred cusp metric** on a compact manifold with boundary M is given by

- A fibration of the boundary: $F \hookrightarrow \partial M \xrightarrow{\phi} B$
- A Riemannian metric on the interior $\mathring{M}$ which in a neighborhood $U \equiv [0, \varepsilon)_x \times \partial M$ of ∂M has the form (with $a \in \mathbb{N}$)

$$g = \frac{dx^2}{x^2} + g_B + x^{2a} g_F$$

$g_B =$ pull-back of a Riemannian metric on B
$g_F =$ a 'fibre metric' for ϕ (i.e. a symmetric 2-tensor on ∂M which restricts to a metric on each fibre of ϕ).

- The boundary ∂M, i.e. $x = 0$, corresponds to 'infinity'
- Fibres collapse as $x \to 0$

Generalizations of fibred cusp metrics

$$g = \frac{dx^2}{x^2} + g_B + x^{2a} g_F$$

- Conformally equivalent: $x^{-2a}g$ (**fibred boundary metrics**), $x^2 g$
- **Multiple fibred cusp metrics:**
  For a *stack* of fibrations $\partial M = B_m \overset{\phi_m}{\to} B_{m-1} \to \cdots \overset{\phi_1}{\to} B_0$

  $$g = \frac{dx^2}{x^2} + g_{B_0} + x^{2a_1} g_{F_1} + \cdots + x^{2a_m} g_{F_m}$$

  g_{B_0} = pull-back of a Riemannian metric on B_0
  g_{F_i} = pull-back of a fibre metric for ϕ_i
  $a_1 < a_2 < \cdots < a_m$ are natural numbers.
- **Iterated fibred cusp/boundary metrics:**
  M = a manifold *with corners*, each boundary hypersurface equipped with a fibration (or stack of fibrations), metric reflects these

Where such metrics arise

- *Locally symmetric spaces:* with $r = -\log x$:

$$g = dr^2 + \phi^* g_B + e^{-2ar} h$$

  or iterated multiple fibred cusp metrics, $M =$ Borel-Serre compactif.
- *Asymptotically conic (AC):* fibred boundary with $F =$ pt and $a = 1$
  E.g.: asympt. Euclidean (AE), asympt. locally Euclidean (ALE):

$$\mathrm{AE} \subset \mathrm{ALE} \subset \mathrm{AC} \subset \{\text{fibred boundary metrics}\}$$

  General relativity (AE); solitons in Ricci flow, gravitational instantons (complete hyper-Kähler 4-manifolds, ALE).
- Quasi-asympt. conic (QAC) $\supset$ QALE: iterated fibred bd. metrics.
  E.g.: natural Kähler metric on the Hilbert scheme $\mathrm{Hilb}_0^n(\mathbb{C}^2)$
- Moduli space of magnetic monopoles: iterated fibred boundary metric.
- Two domains in $\mathbb{R}^n$, tangent to order a: metric of complement is $x^2 g$

Main analytic tool: Pseudodifferential calculus

- ΨDO calculus: systematic tool for analyzing *elliptic operators*
- Also useful for studying *heat kernel*
- $\mathring{M}$ noncompact $\rightsquigarrow$ need methods of singular analysis, e.g. *geometric microlocal analysis* (Melrose)

ΨDO calculus for fibred cusps

Geometric differential operators P for fibred cusp metrics (near ∂M):

$$P = x^{-ka} \sum_{l+|\alpha|+|\beta| \leq k} c_{l\alpha\beta}(x,y,z)(x^{1+a}\partial_x)^l (x^a \partial_y)^\alpha \partial_z^\beta$$

$y \in$ base, $z \in$ fibre, $c_{l\alpha\beta}$ smooth up to $x = 0$.
Degeneration $x^{1+a}\partial_x$, $x^a\partial_y$ $\rightsquigarrow$ need adapted ΨDO calculus.

Ellipticity

P *elliptic* $:\Longleftrightarrow$ (rescaled) principal symbol of P is invertible
P *fully elliptic* $:\Longleftrightarrow$ P is elliptic, and 'normal operators' are invertible

Example: $D_{\mathrm{GB}} = d + d^*$ (Gauß-Bonnet) is elliptic, normal op. $= D_{\mathrm{GB}}^{\text{fibres}}$

State of the art

Small calculus (for fully elliptic ops.): Mazzeo-Melrose, G-Hunsicker,. . .
Large calculus (for elliptic ops.): so far only for a single fibration and under *constant rank assumption* on the boundary operators (Vaillant, G-H)

Objectives: ΨDO calculi

- Construct large calculus for (iterated) multiple fibred cusps, under constant rank assumption
- Construct heat calculus/heat kernel
- Extend to non-constant rank
- Apply to studying mapping properties (asymptotics of solutions, Fredholmness of elliptic operators etc.)

Objectives: Spectral theory, global analysis

- Study analytic continuation of resolvent, generalized eigenfunctions for (iterated) multiple fibred cusp metrics
- Study spectral invariants for these spaces, or for families of compact spaces degenerating to them
- Study L^2 cohomology for these spaces

Selected literature

- Claire Debord, Jean-Marie Lescure, and Frédéric Rochon. Pseudodifferential operators on manifolds with fibred corners. *Ann. Inst. Fourier (Grenoble)*, 65(4):1799–1880, 2015.
- Eugénie Hunsicker and Frédéric Rochon. Weighted Hodge cohomology of iterated fibred cusp metrics. *Ann. Math. Qué.*, 39(2):177–184, 2015.
- Daniel Grieser and Eugénie Hunsicker. Pseudodifferential operator calculus for generalized $\mathbb{Q}$-rank 1 locally symmetric spaces. I. *J. Funct. Anal.*, 257(12):3748–3801, 2009.
- Daniel Grieser and Eugénie Hunsicker. A parametrix construction for the Laplacian on $\mathbb{Q}$-rank 1 locally symmetric spaces. In *Proceedings of the Workshop on Fourier Analysis and Pseudo-Differential Operators, Aalto, Finland*, Trends in Mathematics. Birkhäuser, 2014. 29 pages.
- Tamás Hausel, Eugenie Hunsicker, and Rafe Mazzeo. Hodge cohomology of gravitational instantons. *Duke Math. J.*, 122(3):485–548, 2004.
- Rafe Mazzeo and Richard B. Melrose. Pseudodifferential operators on manifolds with fibred boundaries. *Asian J. Math.*, 2:833–866, 1998.
- Boris Vaillant. *Index and spectral theory for manifolds with generalized fibred cusps*. PhD thesis, Universität Bonn, 2001.

Geometric operators on a class of manifolds with bounded geometry

Bernd Ammann (Regensburg)
Nadine Große (Freiburg)

09/10.11.17, Kick-off Meeting, Potsdam

SPP 2026
GEOMETRY AT INFINITY

Main interests

Study geometric operators P on manifolds with boundary and with bounded geometry.

Main examples: Dirac operator, Laplace-Beltrami operator, Hodge-Laplacian, conformal Laplacian

- Understand L^p-spectra
- Boundary value problems and index theory on compact manifolds with corners and on non-compact complete manifolds with controlled geometry at infinity.
- Understand the relation of the geometry at infinity and spectral properties.

Motivations

- Surgery problems in conformal geometry require to understand the spectrum of

$$P : W^{k,p} \subset L^p(\mathbb{M}_c := \boxed{\mathbb{H}^\ell_c \times \mathbb{S}^n}) \to L^p(\mathbb{M}_c)$$

$k = 1$ Dirac operator
$k = 2$ conformal Laplacian

hyp. space, sec $= -c^2$

conformally interpolates between
$\mathbb{M}_0 = \mathbb{R}^\ell \times \mathbb{S}^n$ and
$\mathbb{M}_1 = \mathbb{R}^{\ell+n} \times (\mathbb{R}^\ell \times \{0\})$

Objectives:

- Solving conformally covariant partial differential equations on $\mathbb{M}_c$.
- Yamabe (type) invariants of $\mathbb{M}_c$, and Yamabe (type) problems on stratified spaces.

Motivations

- Pseudodifferential boundary value problems on domains with *piecewise smooth boundaries*.

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- Spectrum of the Dirac operator and perturbations

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Lie manifolds are complete non-compact manifolds with controlled geometry
– inspired by Melrose's b-calculus for asympt. cyl. manifolds.

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Examples

- asympt. cylindrical manifolds
- asympt. hyperbolic manifolds
- asympt. euclidean manifolds

L^p -spectra for Dirac operators on Lie manifolds

Goal: Determine the L^p-spectrum of the Dirac operator on Lie manifolds with a compatible metric ($\rightsquigarrow$ bounded geometry)

- Note that in general the spectrum depends on p

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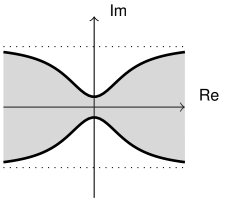
- Note that in general the spectrum depends on p – e.g. for the $\mathbb{H}^{\ell}_{c>0} \times \mathbb{S}^n$ from before:

$$\frac{n}{2} < (\ell - 1)\left|\frac{1}{p} - \frac{1}{2}\right|$$



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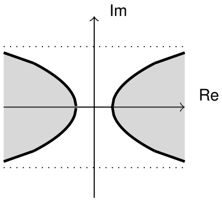

L^p -spectra for Dirac operators on Lie manifolds

Goal: Determine the L^p-spectrum of the Dirac operator on Lie manifolds with a compatible metric ($\rightsquigarrow$ bounded geometry)

Main steps to study

- **Decay estimates** for the Green function $\rightsquigarrow$ parametrix
- **Essential L^p-spectrum for geometric differential operators**
  $\rightsquigarrow$ decomposition principle for L^p-spaces/ Weyl sequences.
- **Relations to the L^p-spectrum of limiting geometries**
- **L^p-index theorems**
  p-dependence of the kernel? Contributions at infinity? On manifolds with boundary?

Spectral density for geometric operators

- *Objective:* Understand essential/absolutely continuous spectra by approximation via disrete spectra

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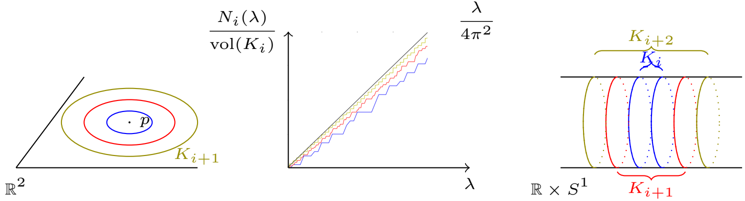
- K_i compact exhaustion of M^n.

  Spectral density $:= \dfrac{d}{d\lambda}\dfrac{N_i(\lambda)}{\text{vol}(K_i)}$

  $N_i(\lambda) := \#\{$eigenv. $\leq \lambda$ of Δ on K_i w.r.t. Dirichlet boundary conditions$\}$

Spectral density for geometric operators

Motivating example:



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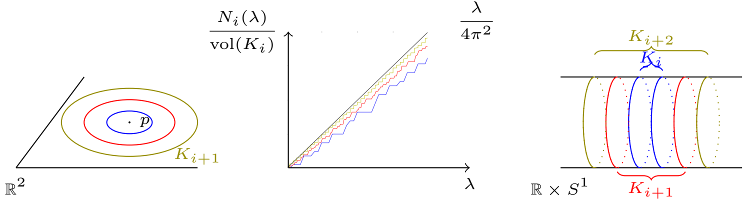

$$\Delta \text{ on } \mathbb{R}^n \quad \overset{\text{Fourier transform}}{\rightsquigarrow} \quad |\xi|^2$$

$$\big(f(\xi), |\xi|^2 g(\xi)\big) = \int_{\mathbb{R}^n} |\xi|^2 f(\xi) g(\xi) d\xi$$

$$\overset{\text{spher. coord.}\ (r=\sqrt{\lambda},\phi)}{=} \int_{\mathbb{R}} \lambda \int_{S^{n-1}} f(\xi) g(\xi) d\omega_{S^{n-1}} \lambda^{n/2-1} d\lambda$$

Spectral density for geometric operators

Main objects of study:

- Existence and (in-)dependence of compact exhaustion
- Different boundary conditions for (Dirac) operators and spectral density
- Use potentials instead of boundary conditions
  - Some boundary conditions fit into this picture, e.g. Dirichlet – infinite potential outside
  - Problem to take into account for Dirac: electrostatic harmonic oscillator potential

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Methods:

- Generalized eigenfunctions $\leftarrow$ decay estimates
- Boundary value problems, perturbations

Thank you for your attention.

Duality and the coarse assembly map

Alexander Engel, Christopher Wulff

University of Regensburg, University of Göttingen

SPP 2026 Kick-Off Meeting, Potsdam, November 2017

SPP 2026
GEOMETRY AT INFINITY

Coarse assembly and co-assembly

Let M be a coarse space, $A \subset M$.

Coarse assembly map (Higson–Roe):

$$\mu : KX_*(M, A) \to K_*(C^*(M, A))$$

Coarse co-assembly map (Emerson–Meyer):

$$\mu^* : K_{1-*}(\mathfrak{c}(M, A)) \to KX^*(M, A)$$

Duality in the sense that there are pairings compatible with μ, μ^*:

$$\langle \mu(x), y \rangle = \langle x, \mu^*(y) \rangle$$

Cup and cap products

Already constructed (W. '16):

- $K_*(\mathfrak{c}(M,A))$ is a ring (cup product $\cup$)
- $K_*(C^*(M,A))$ is a module over $K_*(\mathfrak{c}(M,A))$ (cap product $\cap$)
- if A is a coarse deformation retract of M, a secondary ring structure

$$\Cup : KX^i(M,A) \otimes KX^j(M,A) \to KX^{i+j-1}(M,A)$$

  such that $\mu^* \colon \big(K_{1-*}(\mathfrak{c}(M,A)), \cup\big) \to \big(KX^*(M,A), \Cup\big)$ is a ring homomorphism.

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We plan to complete the picture by constructing a secondary cap product

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and proving that $$\mu(x \Cap \mu^*(y)) = \mu(x) \cap y.$$

Cup and cap products

Already constructed (W. '16):

- $K_*(\mathfrak{c}(M,A))$ is a ring (cup product $\cup$)
- $K_*(C^*(M,A))$ is a module over $K_*(\mathfrak{c}(M,A))$ (cap product $\cap$)
- if A is a coarse deformation retract of M, a secondary ring structure

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Already known (W. '16): Certain coarse index formulas of the form

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Possible further application:

- Coarse Poincaré duality.
- Construct further counterexamples to surjectivity of μ by capping existing ones $x \in K_p(C^*M)$ with classes $c \in K_q(\mathfrak{c}(M))$.

Index theory on Lorentzian manifolds

Institut für Mathematik
Universität Potsdam

9 November 2017

SPP "Geometry at Infinity" kickoff meeting

Potsdam, Germany

Background

Theorem of Bär-Strohmaier in [BS] - *Lorentzian index theorem*:

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- Lorentzian setting, BVPs for Dirac operators by Bär-Hannes in [BH].

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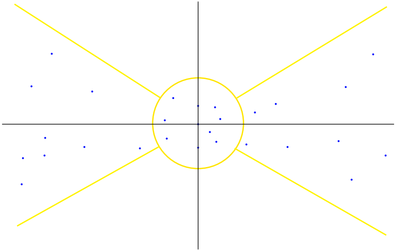
STOP Rarita-Schwinger D does not give rise to a symmetric A.

Via ellipticity, spectrum of A:



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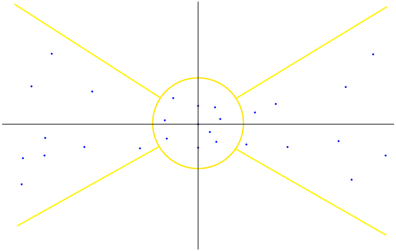

Spectral projectors $\chi^+(A)$ and $\chi^-(A)$,

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- Σ is just a Lipschitz boundary.

References I

[BB] Christian Bär and Werner Ballmann, *Boundary value problems for elliptic differential operators of first order*, Surveys in differential geometry. Vol. XVII, Surv. Differ. Geom., vol. 17, Int. Press, Boston, MA, 2012, pp. 1–78. MR 3076058

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Willmore functional and Lagrangian surfaces

Ernst Kuwert Guofang Wang

University of Freiburg

SPP2026 Kickoff Meeting, Potsdam

Willmore functional and Willmore surfaces

Willmore functional $f : \Sigma \to \mathbb{R}^n$ an immersion

$$\mathcal{W}(f) = \frac{1}{4} \int_\Sigma |H|^2 d\mu_f,$$

The variation formula ($V = \partial_t f$):

$$\frac{d}{dt}\mathcal{W}(f) = \frac{1}{2} \int_\Sigma \langle W(f), V \rangle d\mu_f,$$

$$W(f) \quad = \quad \Delta^\perp H + g^{ik} g^{jl} A^\circ_{ij} \langle A^\circ_{kl}, H \rangle,$$

$f : \Sigma \to \mathbb{R}^n$ Willmore surface, if it is a critical point of W, i.e.

$$\Delta^\perp H + Q(A^\circ) H = 0,$$

where $Q(A^\circ)H = g^{ik} g^{jl} A^\circ_{ij} \langle A^\circ_{kl}, H \rangle$.

There are many inetresting results in the study of Willmore surfaces and Willmore flow. I just mention some of them:

Analytic aspects: Kuwert-Schätzle, Rivière, Kuwert-Li, Lamm, Metzger $\cdots$

Geometric aspects: Bryant, Pinkall, $\cdots$

Willmore conjecture proved by Marques-Neves (2012)

Lagrangian surfaces

Let $\mathbb{C}^2 = \mathbb{R}^4$ with the standard metric, and the symplectic structure $(z_i = x_i + \sqrt{-1}y_i)$

$$\omega = \sum_{i=1,2} dx_i \wedge dy_i.$$

A surface $\Sigma \subset \mathbb{C}^2$ is Lagrangian surface if one of the following equivalent conditions holds:

(1) ω restricted to Σ is zero;

(2) $JT\Sigma = N\Sigma$;

Examples

(a) For example, P$=\{(x_1, x_2, 0, 0)\}$ is a Lagrangian plane.

(b) Whitney sphere: (simplest) Lagrangian immersed sphere,

$$\mathbb{S}^2 \ni (x, y, z) \quad \mapsto \quad \frac{1}{1+z^2}(x(1+iz), y(1+iz)) \in \mathbb{C}^2$$

(c) Clifford torus $\mathbb{S}^1(\sqrt{2}) \times \mathbb{S}^1(\sqrt{2})$ is a Lagrangian surface

(d) Lagrangian graphs: $\{(x_1, x_2, h_1(x_1, x_2), h_2(x_1, x_2))\}$ is Lagrangian, if $\exists\ h : \mathbb{R}^2 \to \mathbb{R}$ with $\nabla h = (h_1, h_2)$.

(e) Cotangent bundle.

By Gromov, only embedded closed Lagrangian surfaces are tori.

Motivation

1. Schoen-Wolfson's work on Lagrangian minimal sufaces (since 2000).

2. Minicozzi (1993) initiated the study of the Willmore functional among Lagrangian immersions. He proved the existence of smooth minimizers of the Willmore functional among closed Lagrangian tori in $\mathbb{C}^2$ and proposed a conjecture.

Lagrangian Willmore Conjecture. *The Willmore energy of a Lagrangian tori in $\mathbb{C}^2$ is at least $2\pi^2$ and the Clifford torus, which is a Lagrangian torus in $\mathbb{C}^2$, minimizes the Willmore energy among Lagrangian tori.*

Previous work und HW Surfaces

1. Kuwert and his collegues' work on Willmore surfaces and Willmore flow.

2. Y. Luo and Wang's work (2015) on Hamiltonian Willmore surfaces and its flow .

Hamiltonian Willmore surface is a critical point of the Willmore functional under Hamiltonian variations, i.e.

$$\mathrm{div}(JW(f)) = 0$$

$$\frac{\partial}{\partial t} f_t = -J\nabla \mathrm{div}\,(JW(f_t)), \quad \forall t \in I.$$

Characterizations for the Whitney sphere Whitney sphere by Castro-Urbano and Ros-Urbano:

$\mathcal{W}(\Sigma) \geq \mathcal{W}(\mathbb{S}_W) = 8\pi$ *for any Lagrangian sphere* Σ. *Equality holds if and only if* Σ *is a Whitney sphere.*

Moreover, t Σ is a Whitney sphere if and only if it holds

$$\breve{A}(X,Y) := A(X,Y) - \frac{1}{n+2}\{g(X,Y)H + g(JX,H)JX + g(JY,H)JY\} = 0,$$

Problem 1. (Optimal rigidity) $\exists\ \delta_0 > 0$, $\forall \delta \in (0, \delta_0)$ *and every Lagrangian sphere* $f \in W^{2,2}_L(\mathbb{S}^2, \mathbb{C}^2)$ *satisfying*

$$\mathcal{W}(f) \leq 8\pi + \delta, \quad \text{or equivalently,} \quad \|\breve{A}\|_{L^2} \leq \delta,$$

$f(\mathbb{S}^2)$ *should be close to a Whitney sphere in a suitable sense?*

De Lellis and Müller: *if* $\int_\Sigma |\mathring{A}|^2$ *is small, then the surface* Σ *is closed to a round sphere in a suitable sense.*

Problem 2. (Rigidity) *Does there exist a constant $\delta_0 > 0$ such that every HW surfacewith $\mathcal{W} \le 8 + \delta$ is a Whitney sphere $\mathbb{S}_W$?*

Problem 3. (Blow-up, Singularities) *Is there a blow-up analysis for HW surfaces? Is the removability of point singularities of HW surfaces true?*

Problem 4. (Blow-up) *Is there a blow-up analysis for the Hamiltonian Willmore flow?*

Problem 5. (Convergence) *Is there a stability result for flow near the Clifford torus? Does the flow with an initial Lagrangian surface near the Whitney sphere converge to a Whitney sphere?*

Problem 6. (Convergence) *Use the Hamiltonian Willmore flow to reprove the Minicozzi's existence result: Among Lagrangian tori there exists a Lagrangian torus with least Willmore functional.*

Thank you very much!

Leibniz Universität Hannover

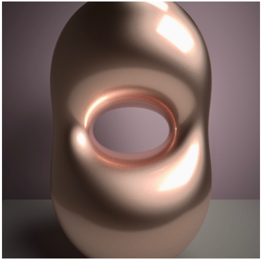
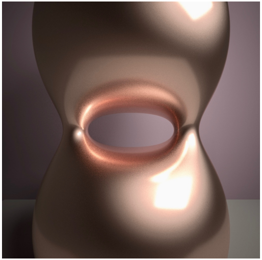
GeometrieWerkstatt

SPP 2026 GEOMETRY AT INFINITY

Minimizers of the Willmore energy with prescribed rectangular conformal class

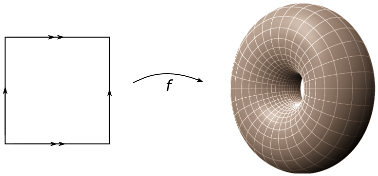
Lynn Heller

Potsdam, November 2017

Problem

Let $f : T^2 \to \mathbb{R}^3,\ \mathbb{S}^3,\ \mathbb{H}^3$ be an immersion



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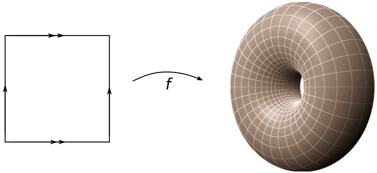

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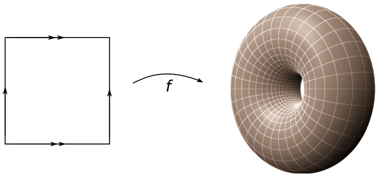

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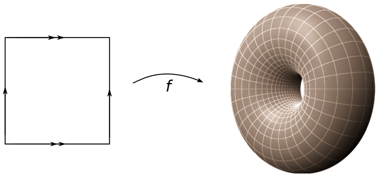

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$$\int_{T^2} (H^2 + \bar{K}) dA = c \int_{T^2} (\kappa_1 - \kappa_2)^2 dA + \text{topological constant}$$

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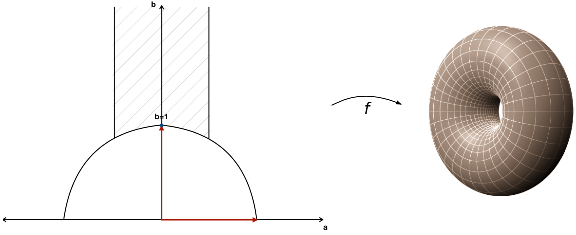

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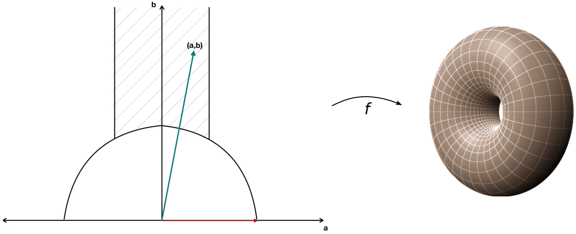

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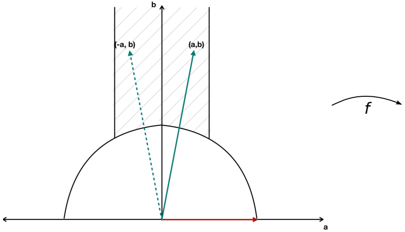

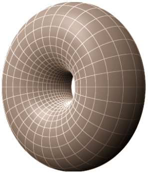

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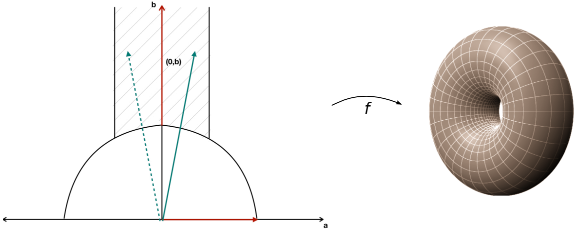

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- The Euler-Lagrange equation is degenerate if and only if the surface is isothermic. For tori this is equivalent to (locally) CMC in a space form [Richter '97].

State of the Art

Existence:

- A minimizer of the CW problem exist if the infimum energy in the conformal class is below 8π [Kuwert-Schätzle, '13]

State of the Art

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- The Clifford torus minimizes the Willmore energy in $\mathbb{R}^3$ for all immersed surfaces of genus $g \geq 1$. [Marques-Neves, '14]
- CW minimizers in an open neighborhood in Teichmüller space near the Clifford torus are equivariant. The minimal energy is continuous but not $\mathcal{C}^1$ at rectangular conformal classes. [H.-Ndiaye, '17]

Candidates

[Kilian - Schmidt -Schmitt '10]

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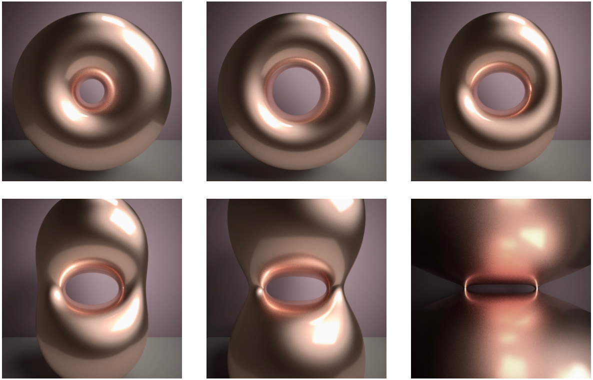

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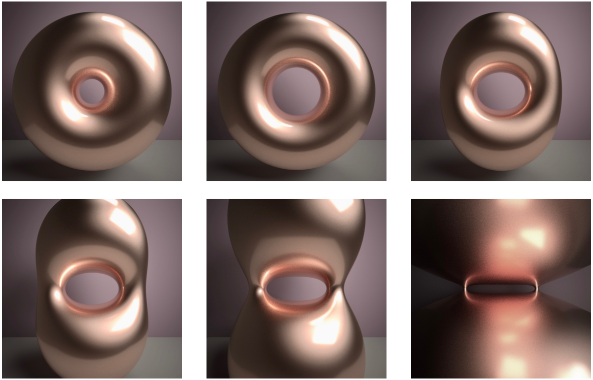

Properties:

- f^b are CMC in $S^3 \rightsquigarrow$ degenerate in the moduli space CW tori

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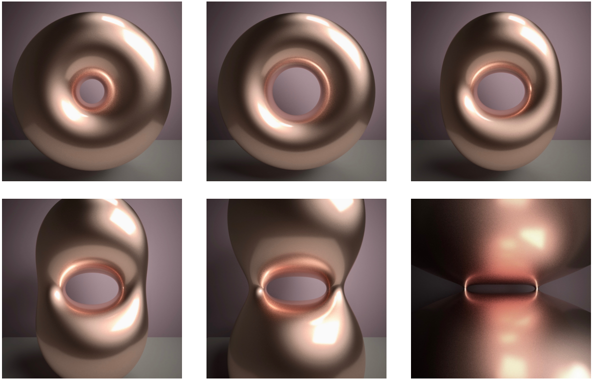

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- f^b are CMC in $S^3 \rightsquigarrow$ degenerate in the moduli space CW tori
- Willmore energy is monotonic in $b \geq 1 \rightsquigarrow$ CW minimizers exist for all rectangular classes.

Strategy

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  $b_0 := \sup\{b \in \mathbb{R}_{\geq 1} \mid f^b \text{ uniquely minimize } \mathcal{W} \text{ for all } (a, b)\}$

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    - f^{b_0} is the unique minimizer for immersions with conformal class (a, b_0)
    - f^{b_0} is $\mathcal{W}$-stable and the kernel is at most 1-dimensional up to invariance

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- Show candidates are unique CW minimizers among isothermic tori.$\rightsquigarrow$Restore the uniqueness of the constrained Willmore minimizer at b_0.

Ramifications

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- Possible generalization to higher codimensions. (In particular codimension 2)

The Willmore energy of degenerating surfaces and singularities of geometric flows

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Beautiful non-orientable surfaces with higher codimension

Let Σ be a closed manifold of dimension two and $f : \Sigma \to \mathbb{R}^n$ an immersion.

- The Willmore functional $\mathcal{W}(f) := \int_\Sigma |H|^2 d\mu$ is conformally invariant.
  $\rightsquigarrow$ particularly interesting from the Calculus of Variations viewpoint
- Minimizers of $\mathcal{W}$ with fixed genus have nice properties.
- What about minimizers with fixed non-orientable genus?

Beautiful non-orientable surfaces with higher codimension

Objectives:

- Prove a lower bound on $\mathcal{W}$ of a closed non-orientable surface at the boundary of moduli space.
- Prove existence of the minimizer of $\mathcal{W}$ with fixed non-orientable genus in higher codimension.
- Study the properties of the minimizers and possible candidates.

Methods: Geometric Analysis, Teichmüller theory, Minimal Surface Theory, Spectral Theory

Singularities of geometric flows with free boundaries

- Consider the *area preserving curve shortening flow* in $\mathbb{R}^2$:

  $\partial_t \gamma = (\kappa - \bar{\kappa})\,\nu,\;$ where $\bar{\kappa}(t) = \frac{\int \kappa ds}{L(\gamma)}$

- Impose *Neumann free boundary conditions*.
- What happens when you let the curve flow?
- How is that different compared to the situation without boundary?

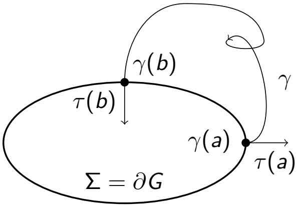

Figure: Geometric situation

Objectives:

- Find a criterion on the initial curve that guarantees the appearance of a singularity in finite time.

  [preprint on ArXiv]

- There seem to be differences compared to the closed case.

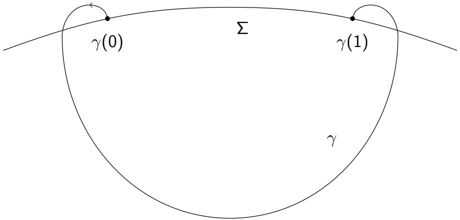

Figure: An initial curve that develops a singularity in finite time.

Singularities of geometric flows with free boundaries

Objectives:

- Based on previous work: There are cases where no singularity develops, the curves *subconverge* to an arc of a circle.

  $\rightsquigarrow$ How is this related to the *(outer) relative Isoperimetric problem*? This problem consists in finding a set attaining the following infimum:

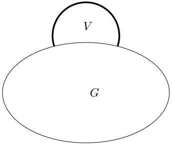

Figure: The relative Isoperimetric problem (outer case)

$$\inf \left\{ \mathcal{H}^1(\partial V \setminus \partial G) : V \subset \mathbb{R}^2 \setminus G \text{ and } |V| = \delta > 0 \right\}$$

Methods: Partial Differential Equations, Blowup Analysis, Geometric Analysis

SPP Project by J. Hirsch

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Existence, regularity and uniqueness results of geometric variational problems

Existence, regularity and uniqueness results of geometric variational problems

Motivation: Investigate the regularity and fine structures of solutions to geometric variational problems.

Methods: geometric measure theory, geometric analysis, partial differential equations and differential geometry

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1. Regularity of $(Q - \frac{1}{2})$-minimisers
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2. Harmonic maps and their homotopy classes
   Principle aim: A Luckhaus' type extension that preserves the homotopy type.
3. Willmore minimising Klein bottle in $\mathbb{R}^3$
   Principle aim: Prove the existence of a minimiser.
4. Uniqueness of conical minimisers
   Principle aim: Provide an example of a minimising harmonic map with a continuum of tangent maps.